

Zero-Bidimension and Various Classes of Bitopological Spaces

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Abstract

The sum theorem and its corollaries are proved for a countable family of zero-dimensional (in the sense of small and large inductive bidimensions) p -closed sets, using a new notion of relative normality whose topological correspondent is also new. The notion of almost n -dimensionality is considered from the bitopological point of view.

Bitopological spaces in which every subset is i -open in its j -closure (i.e., (i, j) -submaximal spaces) are introduced and their properties are studied. Based on the investigations begun in [5] and [14], sufficient conditions are found for bitopological spaces to be $(1, 2)$ -Baire in the class of p -normal spaces. Furthermore, (i, j) - I -spaces are introduced and both the relations between (i, j) -submaximal, (i, j) -nodec and (i, j) - I -spaces, and their properties are studied when two topologies on a set are either independent of each other or interconnected by the inclusion, S -, C - and N -relations or by their combinations.

The final part of the paper deals with the questions of preservation of (i, j) -submaximal and $(2, 1)$ - I -spaces to an image, of D -spaces to an image and an inverse image for both the topological and the bitopological cases. Two theorems are formulated containing, on the one hand, topological conditions and, on the other hand, bitopological ones, under which a topological space is a D -space.

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1. Introduction

Establishment of new results for topological as well as bitopological spaces and strengthening of certain existing and well-known ones have motivated this paper's systematic investigation of different classes of bitopological spaces.

All useful notions have been collected and the following abbreviations are used throughout the paper: TS for a topological space, TsS for a topological subspace, BS for a bitopological space and BsS for a bitopological subspace. The plural form of all abbreviations is 's. Always $i, j \in \{1, 2\}$, $i \neq j$, unless stated otherwise.

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Let (X, τ_1, τ_2) be a BS and \mathcal{P} be some topological property. Then (i, j) - \mathcal{P} denotes the analogue of this property for τ_i with respect to τ_j , and p - \mathcal{P} denotes the conjunction $(1, 2)$ - $\mathcal{P} \wedge (2, 1)$ - \mathcal{P} , that is, p - \mathcal{P} denotes an “absolute” bitopological analogue of \mathcal{P} , where “ p ” is the abbreviation for “pairwise”. Sometimes $(1, 2)$ - $\mathcal{P} \iff (2, 1)$ - \mathcal{P} (and thus $\iff p$ - \mathcal{P}) so that it suffices to consider one of these three bitopological analogues. Moreover, there are certain cases where equivalent topological formulations do not remain equivalent when passing to their bitopological counterparts; in particular, this phenomenon is observed in the case of submaximal spaces [7]. Also note that (X, τ_i) has a property \mathcal{P} if and only if (X, τ_1, τ_2) has a property i - \mathcal{P} , and d - \mathcal{P} is equivalent to 1 - $\mathcal{P} \wedge 2$ - \mathcal{P} , where “ d ” is the abbreviation for “double”.

If τ_1 and τ_2 are independent of each other on X , then along with the properties p - \mathcal{P} and d - \mathcal{P} we can consider the property $\sup \mathcal{P}$, where $\sup \mathcal{P}$ is the \mathcal{P} -property of the TS $(X, \sup(\tau_1, \tau_2))$, clearly not be considered for the case $\tau_1 \subset \tau_2$, that is, in our further notation, for a BS $(X, \tau_1 < \tau_2)$.

The symbol 2^X is used for the power set of the set X , and for a family $\mathcal{A} = \{A_s\}_{s \in S} \subset 2^X$, $\text{co } \mathcal{A}$ denotes the conjugate family $\{X \setminus A_s : A_s \in \mathcal{A}\}_{s \in S}$. If $A \subset X$, then $\tau_i \text{ int } A$ and $\tau_i \text{ cl } A$ denote respectively the interior and the closure of A in the topology τ_i (for a TS (X, τ) the closure of a subset $A \subset X$ is denoted by \bar{A}). A set $A \subset X$ is p -open in (X, τ_1, τ_2) if $A = A_1 \cup A_2$, where $A_i \in \tau_i$. The complement of a p -open set in X is p -closed, that is, $B \subset X$ is p -closed in (X, τ_1, τ_2) if $B = B_1 \cap B_2$, where $B_i \in \text{co } \tau_i$ [11]. Thus, a subset $A \subset X$ is p -open (p -closed) in (X, τ_1, τ_2) if and only if $A = \tau_1 \text{ int } A \cup \tau_2 \text{ int } A$ ($A = \tau_1 \text{ cl } A \cap \tau_2 \text{ cl } A$) and the family of all p -open (p -closed) subsets of a BS (X, τ_1, τ_2) is denoted by $p\text{-}\mathcal{O}(X)$ ($p\text{-}\mathcal{Cl}(X)$). It is clear that $\tau_1 \cup \tau_2 \subset p\text{-}\mathcal{O}(X)$ ($\text{co } \tau_1 \cup \text{co } \tau_2 \subset p\text{-}\mathcal{Cl}(X)$) and so in a BS $(X, \tau_1 < \tau_2)$ we have $p\text{-}\mathcal{O}(X) = \tau_2$ ($p\text{-}\mathcal{Cl}(X) = \text{co } \tau_2$). The notion of a p -open (p -closed) set is equivalent to the notion of a quasi open (quasi closed) set given in [10]. The bitopological boundaries of a subset $A \subset X$ are p -closed sets $(i, j)\text{-Fr } A = \tau_i \text{ cl } A \cap \tau_j \text{ cl } (X \setminus A)$ [11].

Also, to avoid confusion with generally accepted notations, for a BS (X, τ_1, τ_2) we shall use the following double indexation:

$$A_i^d = \{x \in X : x \text{ is an } i\text{-accumulation point of } A\}$$

and

$$A_j^i = \{x \in X : x \text{ is a } j\text{-isolated point of } A\},$$

that is, the lower indices i and j denote the belonging to the topology and, therefore, $i, j \in \{1, 2\}$, while the upper indices d and i are fixed as the accumulation and isolation symbols, respectively; thus $A_j^i = A \setminus A_j^d$, A is a j -discrete set $\iff A = A_j^i$, and

$$\begin{aligned} i\text{-}\mathcal{Bd}(X) &= \{A \in 2^X : \tau_i \text{ int } A = \emptyset\}, \\ i\text{-}\mathcal{D}(X) &= \text{co } i\text{-}\mathcal{Bd}(X) = \{A \in 2^X : \tau_i \text{ cl } A = X\}, \\ i\text{-}\mathcal{DI}(X) &= \{A \in 2^X : A \subset A_i^d\}, \quad (i, j)\text{-}\mathcal{DI}(X) = \{A \in 2^X : A_i^i \subset A_j^d\}, \\ i\text{-}\mathcal{ST}(X) &= \left\{A \in 2^X : A \neq \emptyset, B \in i\text{-}\mathcal{DI}(X) \text{ and } B \subset A \text{ imply } B = \emptyset\right\}, \\ p\text{-}\mathcal{ST}(X) &= \left\{A \in 2^X : A \neq \emptyset, B \in p\text{-}\mathcal{DI}(X) \text{ and } B \subset A \text{ imply } B = \emptyset\right\}, \end{aligned}$$

$$\begin{aligned}
i\text{-}\mathcal{ND}(X) &= \{A \in 2^X : \tau_i \text{ int } \tau_i \text{ cl } A = \emptyset\}, \\
(i, j)\text{-}\mathcal{ND}(X) &= \{A \in 2^X : \tau_i \text{ int } \tau_j \text{ cl } A = \emptyset\}, \\
(i, j)\text{-}\mathcal{Cat}_I(X) &= \\
&= \left\{A \in 2^X : A = \bigcup_{n=1}^{\infty} A_n, A_n \in (i, j)\text{-}\mathcal{ND}(X) \text{ for each } n = \overline{1, \infty}\right\}, \\
(i, j)\text{-}\mathcal{Cat}_{II}(X) &= 2^X \setminus (i, j)\text{-}\mathcal{Cat}_I(X), \\
i\text{-}\mathcal{G}_\delta(X) &= \left\{A \in 2^X : A = \bigcap_{n=1}^{\infty} U_n, U_n \in \tau_i \text{ for each } n = \overline{1, \infty}\right\}
\end{aligned}$$

and

$$i\text{-}\mathcal{F}_\sigma(X) = \text{co } i\text{-}\mathcal{G}_\delta(X) = \left\{A \in 2^X : A = \bigcup_{n=1}^{\infty} F_n, F_n \in \text{co } \tau_i \text{ for each } n = \overline{1, \infty}\right\}$$

are the families of all i -boundary, i -dense, i -dense in themselves, (i, j) -dense in themselves, i -scattered, p -scattered, (i, j) -first category, (i, j) -second category, $i\text{-}\mathcal{G}_\delta$ and $i\text{-}\mathcal{F}_\sigma$ -subsets of X , respectively; note also here, that a subset A of a BS (X, τ_1, τ_2) is of (i, j) -first (second) category, i.e., A is of $(i, j)\text{-}\mathcal{Cat}_I$ ($(i, j)\text{-}\mathcal{Cat}_{II}$) if it is of (i, j) -first (second) category in itself [15].

Definition 1.1. Let (X, τ_1, τ_2) be a BS. Then

- (1) (X, τ_1, τ_2) is $R\text{-}p\text{-}T_1$ (i.e., $p\text{-}T_1$ in the sense of Reilly) if it is $d\text{-}T_1$ [19].
- (2) (X, τ_1, τ_2) is (i, j) -regular if for each point $x \in X$ and each i -closed set $F \subset X$, $x \notin F$, there exist an i -open set U and a j -open set V that $x \in U$, $F \subset V$ and $U \cap V = \emptyset$ [16].
- (3) (X, τ_1, τ_2) is p -normal if for every pair of disjoint sets A, B in X , where A is 1-closed and B is 2-closed, there exist a 2-open set U and a 1-open set V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$ [16].

Moreover, (X, τ_1, τ_2) is hereditarily p -normal if every one of its BSs is p -normal [11].

- (4) (X, τ_1, τ_2) is p -connected if X cannot be expressed as a union of two disjoint sets A and B such that $A \in \tau_1 \setminus \{\emptyset\}$ and $B \in \tau_2 \setminus \{\emptyset\}$ [18] (see also [6], [8], [17]).
- (5) (X, τ_1, τ_2) is (i, j) -extremally disconnected if $\tau_j \text{ cl } U = \tau_i \text{ int } \tau_j \text{ cl } U$ for every set $U \in \tau_i$ or, equivalently, if $\tau_j \text{ cl } \tau_i \text{ int } A = \tau_i \text{ int } \tau_j \text{ cl } \tau_i \text{ int } A$ for every subset $A \subset X$ [3].

Evidently, (X, τ_1, τ_2) is $(1, 2)$ -extremally disconnected $\iff (X, \tau_1, \tau_2)$ is $(2, 1)$ -extremally disconnected $\iff (X, \tau_1, \tau_2)$ is p -extremally disconnected.

- (6) (X, τ_1, τ_2) is an (i, j) -Baire space or an almost (i, j) -Baire space (briefly, $(i, j)\text{-BrS}$ or $A\text{-}(i, j)\text{-BrS}$) if every nonempty i -open subset of X is of (i, j) -second category or of (i, j) -second category in X [13], [15].
- (7) (X, τ_1, τ_2) is (i, j) -nodec if its every (i, j) -nowhere dense subset is j -closed and i -discrete [14].

Furthermore, in a BS (X, τ_1, τ_2)

- (8) τ_1 is coupled to τ_2 (briefly, $\tau_1 C \tau_2$) if $\tau_1 \text{ cl } U \subseteq \tau_2 \text{ cl } U$ for every set $U \in \tau_1$ or, equivalently, if $\tau_1 \text{ cl } \tau_1 \text{ int } A \subseteq \tau_2 \text{ cl } \tau_1 \text{ int } A$ for every subset $A \subset X$ [22].
- (9) τ_1 is near τ_2 (briefly, $\tau_1 N \tau_2$) if $\tau_1 \text{ cl } U \subseteq \tau_2 \text{ cl } U$ for every set $U \in \tau_2$ or, equivalently, if $\tau_1 \text{ cl } \tau_2 \text{ int } A \subseteq \tau_2 \text{ cl } \tau_2 \text{ int } A$ for every subset $A \subset X$ [15].
- (10) τ_1 and τ_2 are S -related on X (briefly, $\tau_1 S \tau_2$) if $\tau_1 \text{ int } A \subset \tau_1 \text{ cl } \tau_2 \text{ int } A \wedge \tau_2 \text{ int } A \subset \tau_2 \text{ cl } \tau_1 \text{ int } A$ for every subset $A \subset X$ [20].

Since for a BS $(X, \tau_1 < \tau_2)$ the inclusions

$$\begin{array}{ccc} (2, 1)\text{-}\mathcal{ND}(X) & \subset & 2\text{-}\mathcal{ND}(X) \\ \cap & & \cap \\ 1\text{-}\mathcal{ND}(X) & \subset & (1, 2)\text{-}\mathcal{ND}(X) \end{array}$$

are correct, in the case where $\tau_1 \subset \tau_2$ we come to the following evident implications:

$$(X, \tau_1, \tau_2) \text{ is 1-nodec} \implies (X, \tau_1, \tau_2) \text{ is } (2, 1)\text{-nodec}$$

and

$$(X, \tau_1, \tau_2) \text{ is } (1, 2)\text{-nodec} \implies (X, \tau_1, \tau_2) \text{ is 2-nodec}.$$

Moreover, according to (1) of Theorem 2.1.10 in [15], for a BS $(X, \tau_1 <_S \tau_2)$, where $\tau_1 <_S \tau_2 \iff (\tau_1 \subset \tau_2 \wedge \tau_1 S \tau_2)$, in addition to the above implications, we have:

$$\begin{array}{ccc} (X, \tau_1, \tau_2) \text{ is 1-nodec} & \implies & (X, \tau_1, \tau_2) \text{ is } (2, 1)\text{-nodec} \\ \Downarrow & & \Downarrow \\ (X, \tau_1, \tau_2) \text{ is } (1, 2)\text{-nodec} & \implies & (X, \tau_1, \tau_2) \text{ is 2-nodec}. \end{array}$$

Remark 1.2. Every BS (Y, τ'_1, τ'_2) of a $(1, 2)$ -nodec BS $(X, \tau_1 < \tau_2)$ is also $(1, 2)$ -nodec. Indeed: if $A \in (1, 2)\text{-}\mathcal{ND}(Y)$, then by (1) of Theorem 1.5.13 in [15], $A \in (1, 2)\text{-}\mathcal{ND}(X)$ and, hence, $A = \tau_2 \text{ cl } A = A_1^i$. Thus A is 2-closed and 1-discrete in (Y, τ'_1, τ'_2) too.

Definition 1.3. A subset A of a BS (X, τ_1, τ_2) is said to be (i, j) -locally closed if for each point $x \in A$ there exists a set $U \in \tau_i$ such that $x \in U$ and $U \cap A = U \cap \tau_j \text{ cl } A$ [15].

The families of all such subsets of X are denoted by $(i, j)\text{-}\mathcal{LC}(X)$ and it is not difficult to see that

$$\begin{aligned} A \in (i, j)\text{-}\mathcal{LC}(X) &\iff (A \in \tau'_i \text{ in the BS } (\tau_j \text{ cl } A, \tau'_1, \tau'_2)) \iff \\ &\iff (A = U \cap F, \text{ where } U \in \tau_i \text{ and } F \in \text{co } \tau_j). \end{aligned}$$

Hence $\tau_i \cup \text{co } \tau_j \subset (i, j)\text{-}\mathcal{LC}(X)$ and for a BS $(X, \tau_1 < \tau_2)$ the following inclusions hold:

$$\begin{array}{ccccc} \tau_2 \cup \text{co } \tau_2 & \subset & 2\text{-}\mathcal{LC}(X) & \supset & (2, 1)\text{-}\mathcal{LC}(X) \supset \tau_2 \cup \text{co } \tau_1 \\ & & \cup & & \cup \\ \tau_1 \cup \text{co } \tau_2 & \subset & (1, 2)\text{-}\mathcal{LC}(X) & \supset & 1\text{-}\mathcal{LC}(X) \supset \tau_1 \cup \text{co } \tau_1. \end{array}$$

Definition 1.4. Let (x, A) be a pair in a BS (X, τ_1, τ_2) such that $A \in \text{co } \tau_i$ and $x \notin A$. Then a p -closed set T is a partition, corresponding to the pair (x, A) , if $X \setminus T = H_1 \cup H_2$, where $H_i \in \tau_i \setminus \{\emptyset\}$, $x \in H_i$, $A \subset H_j$ and $H_1 \cap H_2 = \emptyset$.

If (x, A) is a pair in the above sense, then one can easily verify that for an i -open neighborhood $U(x)$ (j -open neighborhood $U(A)$) such that $\tau_j \text{ cl } U(x) \subset X \setminus A$ ($\tau_i \text{ cl } U(A) \subset X \setminus \{x\}$), the sets $(j, i)\text{-Fr } U(x)$ ($(i, j)\text{-Fr } U(A)$) are partitions, corresponding to (x, A) and, conversely, if T is a partition, corresponding to (x, A) , then $(j, i)\text{-Fr } H_i \subset T$.

For the pairwise small inductive dimension $p\text{-ind } X$ we have:

$p\text{-ind } X = 0 \iff (\tau_1 \text{ has a base consisting of 2-closed sets and } \tau_2 \text{ has a base consisting of 1-closed sets}) \iff (\text{the empty set is a partition corresponding to any pair } (x, A), \text{ where } A \in \text{co } \tau_1, x \notin A \text{ and the empty set is a partition corresponding to any pair } (x, A), \text{ where } A \in \text{co } \tau_2, x \notin A) \text{ [11], [12].}$

Definition 1.5. Let (A, B) be a pair of subsets of a BS (X, τ_1, τ_2) such that $A \in \text{co } \tau_1$, $B \in \text{co } \tau_2$ and $A \cap B = \emptyset$. Then we say that a p -closed set T is a partition, corresponding to (A, B) , if $X \setminus T = H_1 \cup H_2$, where $H_i \in \tau_i \setminus \{\emptyset\}$, $A \subset H_2$, $B \subset H_1$ and $H_1 \cap H_2 = \emptyset$.

If (A, B) is a pair in the above sense and there exists a 2-open neighborhood $U(A)$ (1-open neighborhood $U(B)$) such that $\tau_1 \text{ cl } U(A) \subset X \setminus B$ ($\tau_2 \text{ cl } U(B) \subset X \setminus A$), then $(1, 2)\text{-Fr } U(A)$ ($(2, 1)\text{-Fr } U(B)$) is a partition, corresponding to (A, B) and, conversely, if T is a partition, corresponding to (A, B) , then $(j, i)\text{-Fr } H_i \subset T$.

Now, for the pairwise large inductive dimension $p\text{-Ind } X$ we have:

$p\text{-Ind } X = 0 \iff (\text{the empty set is a partition corresponding to any pair } (A, B), \text{ where } A \in \text{co } \tau_1, B \in \text{co } \tau_2 \text{ and } A \cap B = \emptyset) \iff (\text{for every 1-closed set } F \text{ and any 2-neighborhood } U(F) \text{ there exists a neighborhood } V(F) \in \tau_2 \cap \text{co } \tau_1 \text{ such that } V(F) \subset U(F) \text{ and, for every 2-closed set } \Phi \text{ and any 1-neighborhood } U(\Phi) \text{ there exists a neighborhood } V(\Phi) \in \tau_1 \cap \text{co } \tau_2 \text{ such that } V(\Phi) \subset U(\Phi)) \text{ [11], [12].}$

Definition 1.6. Let (X, τ_1, τ_2) and (Y, γ_1, γ_2) be BS's. Then a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \gamma_1, \gamma_2)$ is said to be i -continuous (i -open, i -closed) if the induced functions $f : (X, \tau_i) \rightarrow (Y, \gamma_i)$ are continuous (open, closed) [18].

Finally, please note that all bitopological generalizations are constructed in the commonly accepted manner, so that if the topologies coincide, we obtain the original topological notions.

2. Some Special Notions and the Sum Theorem for Bidimension Zero

Our first step in this section is to prove some simple, but nevertheless principal facts, having an independent interest.

Proposition 2.1. For any subsets $A \in \tau_2$ ($A \in \text{co } \tau_1$) and $B \in \tau_1$ ($B \in \text{co } \tau_2$) of a BS (X, τ_1, τ_2) the sets $A \setminus B$ and $B \setminus A$ are p -separated, that is,

$$(\tau_1 \text{ cl}(A \setminus B) \cap (B \setminus A)) \cup ((A \setminus B) \cap \tau_2 \text{ cl}(B \setminus A)) = \emptyset.$$

Proof. We will prove the condition, given without brackets, since the other one can be proved similarly.

Evidently, $A = A \cap (A \cup B) \in \tau'_2$ and $B = B \cap (A \cup B) \in \tau'_1$ in $(A \cup B, \tau'_1, \tau'_2)$. Therefore $A \setminus B = (A \cup B) \setminus B \in \text{co } \tau'_1$ and $B \setminus A = (A \cup B) \setminus A \in \text{co } \tau'_2$. Let us consider the BsS $((A \setminus B) \cup (B \setminus A), \tau''_1, \tau''_2)$ of the BS $(A \cup B, \tau'_1, \tau'_2)$. Then

$$(A \setminus B) \cap ((A \setminus B) \cup (B \setminus A)) = A \setminus B \in \text{co } \tau''_1$$

and

$$(B \setminus A) \cap ((A \setminus B) \cup (B \setminus A)) = B \setminus A \in \text{co } \tau''_2.$$

Hence $A \setminus B \in \tau''_2$ and $B \setminus A \in \tau''_1$ so that

$$(\tau''_1 \text{ cl}(A \setminus B) \cap (B \setminus A)) \cup ((A \setminus B) \cap \tau''_2 \text{ cl}(B \setminus A)) = \emptyset$$

and, thus,

$$(\tau_1 \text{ cl}(A \setminus B) \cap (B \setminus A)) \cup ((A \setminus B) \cap \tau_2 \text{ cl}(B \setminus A)) = \emptyset. \quad \square$$

Definition 2.2. Let (Y, τ'_1, τ'_2) and (Z, τ''_1, τ''_2) be BsS's of a BS (X, τ_1, τ_2) such that $X = Y \cup Z$. Then the (i, j) -closures of any subset $A \subset X$, corresponding to the pair (Y, Z) , are the sets

$$(i, j)\text{-cl } A(Y, Z) = \tau_i \text{ cl}(A \cap Y) \cup \tau_j \text{ cl}(A \cap (Z \setminus Y)).$$

Moreover,

$$D\text{-cl } A(Y, Z) = ((1, 2)\text{-cl } A(Y, Z) \cap Y) \cup ((2, 1)\text{-cl } A(Z, Y) \cap Z)$$

and A is D -closed with respect to (Y, Z) if $A = D\text{-cl } A(Y, Z)$.

Proposition 2.3. *If for a BS (X, τ_1, τ_2) there are BsS's (Y, τ'_1, τ'_2) , (Z, τ''_1, τ''_2) such that $X = Y \cup Z$ and the sets $Y \setminus Z$, $Z \setminus Y$ are p -separated, then for every subset $A \subset X$ we have*

$$D\text{-cl } A(Y, Z) = \tau'_1 \text{ cl}(A \cap Y) \cup \tau''_2 \text{ cl}(A \cap Z).$$

Proof. Since $\tau_2 \text{ cl}(Z \setminus Y) \cap (Y \setminus Z) = \emptyset$, we have $\tau_2 \text{ cl}(Z \setminus Y) \subset Z$ so that

$$\tau_2 \text{ cl}(A \cap (Z \setminus Y)) \cap Y \subset \tau_2 \text{ cl}(A \cap (Z \setminus Y)) \subset \tau_2 \text{ cl}(A \cap Z) \cap Z.$$

Hence

$$(\tau_1 \text{ cl}(A \cap Y) \cup \tau_2 \text{ cl}(A \cap (Z \setminus Y))) \cap Y = (\tau_1 \text{ cl}(A \cap Y) \cap Y) \cup M,$$

where

$$M = \tau_2 \text{ cl}(A \cap (Z \setminus Y)) \cap Y \subset \tau_2 \text{ cl}(A \cap Z) \cap Z.$$

By the similar manner we obtain that

$$(\tau_2 \text{ cl}(A \cap Z) \cup \tau_1 \text{ cl}(A \cap (Y \setminus Z))) \cap Z = (\tau_2 \text{ cl}(A \cap Z) \cap Z) \cup N,$$

where

$$N = \tau_1 \text{ cl}(A \cap (Y \setminus Z)) \cap Z \subset \tau_1 \text{ cl}(A \cap Y) \cap Y.$$

Thus

$$\begin{aligned} & \left((\tau_1 \text{ cl}(A \cap Y) \cup \tau_2 \text{ cl}(A \cap (Z \setminus Y))) \cap Y \right) \cup \\ & \cup \left((\tau_2 \text{ cl}(A \cap Z) \cup \tau_1 \text{ cl}(A \cap (Y \setminus Z))) \cap Z \right) = \\ & = (\tau_1 \text{ cl}(A \cap Y) \cap Y) \cup (\tau_2 \text{ cl}(A \cap Z) \cap Z) = \tau_1' \text{ cl}(A \cap Y) \cap \tau_2'' \text{ cl}(A \cap Z). \quad \square \end{aligned}$$

Corollary 2.4. *If (Y, τ_1', τ_2') and (Z, τ_1'', τ_2'') are BsS's of a BS (X, τ_1, τ_2) , where $Y \in \tau_2$ ($Y \in \text{co } \tau_1$), $Z \in \tau_1$ ($Z \in \text{co } \tau_2$) and $X = Y \cup Z$, then*

$$D\text{-cl } A(Y, Z) = \tau_1' \text{ cl}(A \cap Y) \cup \tau_2'' \text{ cl}(A \cap Z).$$

Proof. By Proposition 2.1, the sets $Y \setminus Z$ and $Z \setminus Y$ are p -separated and, hence, it remains to use Proposition 2.3. \square

Corollary 2.5. *If (Y, τ_1', τ_2') and (Z, τ_1'', τ_2'') are BsS's of a BS (X, τ_1, τ_2) , $X = Y \cup Z$ and $Y \setminus Z$, $Z \setminus Y$ are p -separated, then $A = D\text{-cl } A(Y, Z)$ if and only if $A \cap Y \in \text{co } \tau_1'$ and $A \cap Z \in \text{co } \tau_2''$.*

Proof. If $A = D\text{-cl } A(Y, Z)$, then according to Proposition 2.3,

$$A = \tau_1' \text{ cl}(A \cap Y) \cup \tau_2'' \text{ cl}(A \cap Z) = (A \cap Y) \cup (A \cap Z)$$

and so $A \cap Y \in \text{co } \tau_1'$, $A \cap Z \in \text{co } \tau_2''$. Conversely, if $A \cap Y \in \text{co } \tau_1'$ and $A \cap Z \in \text{co } \tau_2''$, then once more applying Proposition 2.3, we obtain that

$$\begin{aligned} D\text{-cl } A(Y, Z) &= \tau_1' \text{ cl}(A \cap Y) \cup \tau_2'' \text{ cl}(A \cap Z) = \\ &= (A \cap Y) \cup (A \cap Z) = A \cap (Y \cup Z) = A \cap X = A. \quad \square \end{aligned}$$

Corollary 2.6. *If (Y, τ_1', τ_2') and (Z, τ_1'', τ_2'') are BsS's of a BS (X, τ_1, τ_2) such that $X = Y \cup Z$. Then for every subset $A \subset X$ the following conditions hold:*

- (1) $A \subset (1, 2)\text{-cl } A(Y, Z) \cap (2, 1)\text{-cl } A(Z, Y)$.
- (2) $(1, 2)\text{-cl } A(Y, Z) \cup (2, 1)\text{-cl } A(Z, Y) = \tau_1 \text{ cl}(A \cap Y) \cup \tau_2 \text{ cl}(A \cap Z)$.
- (3) *If $Y \cap Z = \emptyset$, then*

$$(1, 2)\text{-cl } A(Y, Z) = (2, 1)\text{-cl } A(Z, Y) = \tau_1 \text{ cl}(A \cap Y) \cup \tau_2 \text{ cl}(A \cap Z).$$

- (4) *If $Y = \emptyset$, that is, if $X = Z$, then*

$$(1, 2)\text{-cl } A(\emptyset, Z) = \tau_2 \text{ cl } A = (2, 1)\text{-cl } A(Z, \emptyset).$$

- (5) *If $Z = \emptyset$, that is, if $X = Y$, then*

$$(1, 2)\text{-cl } A(Y, \emptyset) = \tau_1 \text{ cl } A = (2, 1)\text{-cl } A(\emptyset, Y).$$

(6) If $A \subset Y$, then $(1, 2)\text{-cl } A(Y, Z) = \tau_1 \text{ cl } A$ and if $A \subset Y \setminus Z$, then

$$(1, 2)\text{-cl } A(Y, Z) = \tau_1 \text{ cl } A = (2, 1)\text{-cl } A(Z, Y).$$

(7) If $A \subset Z$, then $(2, 1)\text{-cl } A(Z, Y) = \tau_2 \text{ cl } A$ and if $A \subset Z \setminus Y$, then

$$(1, 2)\text{-cl } A(Y, Z) = \tau_2 \text{ cl } A = (2, 1)\text{-cl } A(Z, Y).$$

(8) If $Y = Z = X$, then

$$(1, 2)\text{-cl } A(X, X) = \tau_1 \text{ cl } A, \quad (2, 1)\text{-cl } A(X, X) = \tau_2 \text{ cl } A$$

and

$$D\text{-cl } A(X, X) = \tau_1 \text{ cl } A \cup \tau_2 \text{ cl } A.$$

(9) If $Y = A$, $Z = X \setminus A$, then

$$(i, j)\text{-Fr } A = (i, j)\text{-cl } A(A, X \setminus A) \cap (j, i)\text{-cl } (X \setminus A)(X \setminus A, A).$$

Moreover, if $\tau_1 \subset \tau_2$, then

(10) $\tau_2 \text{ cl } A \subset (1, 2)\text{-cl } A(Y, Z) \cap (2, 1)\text{-cl } A(Z, Y) \subset \tau_1 \text{ cl } A$.

(11) If $A \subset Y$, then $\tau_2 \text{ cl } A \subset (2, 1)\text{-cl } A(Z, Y) \subset (1, 2)\text{-cl } A(Y, Z) = \tau_1 \text{ cl } A$.

(12) If $A \subset Z$, then $\tau_2 \text{ cl } A = (2, 1)\text{-cl } A(Z, Y) \subset (1, 2)\text{-cl } A(Y, Z) \subset \tau_1 \text{ cl } A$.

(13) $D\text{-cl } A(X, X) = \tau_1 \text{ cl } A$.

The proof of (1)–(13) is trivial.

Clearly, the (i, j) -closures and D -closure have many other interesting properties.

Proposition 2.7. For a BS (X, τ_1, τ_2) the following conditions are equivalent:

- (1) (X, τ_1, τ_2) is hereditarily p -normal.
- (2) Every p -open subset of X is p -normal.
- (3) If A, B are p -separated subsets of X , then there are sets $U \in \tau_2$, $V \in \tau_1$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

Proof. The equivalence (1) \iff (3) is proved in Theorem 0.2.2 from [15]. The implication (1) \implies (2) is obvious. The proof of the implication (2) \implies (3) is given in the first part of the proof of Theorem 0.2.2 from [15], since the set $Y = X \setminus (\tau_1 \text{ cl } A \cap \tau_2 \text{ cl } B)$ is p -open. \square

Corollary 2.8. If a BS $(X, \tau <_C \tau_2)$ is hereditarily p -normal, then it is hereditarily 1-normal.

Proof. By the well-known topological fact it is sufficient to prove only that every 1-open BsS of X is 1-normal. Let $U \in \tau_1 \setminus \{\emptyset\}$ be any set. Then $U \in p\text{-}\mathcal{O}(X)$ and by (2) of Proposition 2.7, (U, τ'_1, τ'_2) is p -normal. On the other hand, by (2) of

Corollary 2.2.8 in [15], $\tau'_1 <_C \tau'_2$ and, hence, by (4) of the same corollary, (U, τ'_1, τ'_2) is 1-normal. \square

Corollary 2.9. *If a BS $(X, \tau <_N \tau_2)$ is hereditarily 2-normal, then it is hereditarily p -normal and so it is hereditarily 1-normal.*

Proof. When $\tau_1 <_N \tau_2$, then also $\tau_1 <_C \tau_2$ (Corollary 2.3.10 in [15]) and by Corollary 2.8 it suffices to prove only that every hereditarily 2-normal BS $(X, \tau <_N \tau_2)$ is hereditarily p -normal. Let $U \in p\text{-}\mathcal{O}(X)$ be any set. Then $U \in \tau_2$ as $\tau_1 \subset \tau_2$, so that (U, τ'_1, τ'_2) is 2-normal. But, according to (2) of Corollary 2.2.13 in [15], $\tau'_1 <_N \tau'_2$ and, hence, by (4) of the same corollary, (U, τ'_1, τ'_2) is p -normal. \square

Definition 2.10. A BS (X, τ_1, τ_2) is p -hypernormal if for every pair A, B of p -separated subsets of X (i.e., $(\tau_1 \text{ cl } A \cap B) \cup (A \cap \tau_2 \text{ cl } B) = \emptyset$) there are sets $U \in \tau_2, V \in \tau_1$ such that $A \subset U, B \subset V$ and $\tau_1 \text{ cl } U \cap \tau_2 \text{ cl } V = \emptyset$.

Take place the following

Proposition 2.11. *A BS (X, τ_1, τ_2) is p -hypernormal if and only if it is p -extremally disconnected and hereditarily p -normal.*

Proof. Evidently, every p -extremally disconnected and hereditarily p -normal BS is p -hypernormal. Therefore, suppose that (X, τ_1, τ_2) is p -hypernormal. Then, by Proposition 2.7, (X, τ_1, τ_2) is hereditarily p -normal and so, by (5) of Definition 1.1, it remains to prove only that $\tau_j \text{ cl } U \in \tau_i$ for every i -open subset $U \subset X$. Since $U \in \tau_i$, we have

$$(U \cap \tau_i \text{ cl } (X \setminus \tau_j \text{ cl } U)) \cup (\tau_j \text{ cl } U \cap (X \setminus \tau_j \text{ cl } U)) = \emptyset$$

so that U and $X \setminus \tau_j \text{ cl } U$ are p -separated and, hence, by condition, there are sets $M \in \tau_i, N \in \tau_j$ such that $U \subset M, (X \setminus \tau_j \text{ cl } U) \subset N$ and $\tau_j \text{ cl } M \cap \tau_i \text{ cl } N = \emptyset$. Since $\tau_j \text{ cl } U \subset \tau_j \text{ cl } M$ and $\tau_i \text{ cl } (X \setminus \tau_j \text{ cl } U) \subset \tau_i \text{ cl } N$, we obtain that $\tau_j \text{ cl } U \cap \tau_i \text{ cl } (X \setminus \tau_j \text{ cl } U) = \emptyset$ so that

$$\tau_j \text{ cl } U \cap (X \setminus \tau_i \text{ int } \tau_j \text{ cl } U) = \emptyset$$

and, thus, $\tau_j \text{ cl } U = \tau_i \text{ int } \tau_j \text{ cl } U$. \square

Corollary 2.12. *If (X, τ_1, τ_2) is p -hypernormal, then*

- (1) $p\text{-Ind } X = 0$ or, equivalently, $p\text{-dim } X = 0$.
- (2) *If (X, τ_1, τ_2) is R- $p\text{-T}_1$, then*

$$p\text{-ind } X = p\text{-Ind } X = p\text{-dim } X = 0.$$

Moreover, in addition to the above conditions, for a BS $(X, \tau_1 <_C \tau_2)$ we have:

- (3) (X, τ_1, τ_2) is 1-hypernormal.
- (4) $1\text{-Ind } X = 1\text{-dim } X = p\text{-Ind } X = p\text{-dim } X = 0$
and if (X, τ_1, τ_2) is 1- T_1 , then

$$1\text{-Ind } X = 1\text{-dim } X = p\text{-Ind } X = p\text{-dim } X = d\text{-ind } X = p\text{-ind } X = 0.$$

Proof. (1) The equality $p\text{-Ind } X = 0$ is obvious, since, by Proposition 2.11, (X, τ_1, τ_2) is p -normal and p -extremally disconnected. Hence, it remains to use (1) of Corollary 3.3.5 in [15].

(2) It suffices to use (2) of Corollary 3.3.5 in [15].

(3) According to Corollary 2.8, (X, τ_1, τ_2) is hereditarily 1-normal, and by (8) of Corollary 2.2.8 in [15], (X, τ_1, τ_2) is 1-extremally disconnected.

(4) The first part follows directly from the well-known topological fact, (1) above and (1) of Theorem 3.2.38 in [15]. The rest is an immediate consequence of (2) above and (5) of Theorem 3.1.36 in [15].

Corollary 2.13. *For a BS $(X, \tau_1 <_N \tau_2)$ the following implications hold:*

$$\begin{aligned} (X, \tau_1, \tau_2) \text{ is } d\text{-hypernormal} &\iff (X, \tau_1, \tau_2) \text{ is } 2\text{-hypernormal} \implies \\ \implies (X, \tau_1, \tau_2) \text{ is } p\text{-hypernormal} &\implies (X, \tau_1, \tau_2) \text{ is } 1\text{-hypernormal}. \end{aligned}$$

Proof. If (X, τ_1, τ_2) is 2-hypernormal, then by Corollary 2.9, it is hereditarily p -normal, and by (7) of Corollary 2.3.13 in [15], it is also p -extremally disconnected. Hence, according to Proposition 2.11, (X, τ_1, τ_2) is p -hypernormal. Furthermore, by Corollary 2.3.10 in [15], $\tau_1 <_C \tau_2$ and so, by (3) of Corollary 2.12, (X, τ_1, τ_2) is 1-hypernormal.

Now, the first implication from right to left is obvious. \square

The rest of this section is devoted to new notions of relative normality of BS's and their applications in the theory of dimension of BS's. Their topological counterpart is new as well. As will be shown below, these notions prove to be the key tool for correcting an error made in proving Theorem 3.2.26 and its Corollaries 3.2.27–3.2.30 in [15]. In particular, it allows us to prove the above-mentioned theorem and its corollaries for a sequence of p -closed sets. The latter circumstance emphasizes once more a special role which relative (bi)topological properties play not only in the development of respective theories, but also in the strengthening of the previously known results.

Definition 2.14. We will say that a TsS (Y, τ') of a TS (X, τ) is WS-supernormal in X if for each pair of disjoint sets A, B , where A is closed in (Y, τ') and B is closed in (X, τ) , there are disjoint sets U, V such that U is open in (Y, τ') , V is open in (X, τ) and $A \subset U, B \subset V$.

It is clear that if (X, τ) is normal, then every closed subspace of (X, τ) is WS-normal in X , and if (Y, τ') is WS-supernormal in X , then (Y, τ') is normal (in itself). But for the opposite implication we have the following elementary

Example 2.15. Let us consider the natural TS (\mathbb{R}, ω) and an open interval $(a, b) \subset \mathbb{R}$, which is normal (in itself). If $A = [c, b)$ and $B = [b, d]$ are sets, where $a < c < b < d$, then $A \in \text{co } \omega'$ in $((a, b), \omega')$, $B \in \text{co } \omega$ and $A \cap B = \emptyset$. But for any neighborhood $V(B) \in \omega$ we have $A \cap V(B) \neq \emptyset$.

Now, we give the bitopological modifications of relative WS-supernormality in the above sense and of relative strong normality in the sense of [4].

Definition 2.16. We will say that a BsS (Y, τ'_1, τ'_2) of a BS (X, τ_1, τ_2) is (i, j) -WS-supernormal (p -strongly normal) in X if for each pair of disjoint sets $A \in \text{co } \tau'_i$, $B \in \text{co } \tau'_j$ ($A \in \text{co } \tau'_1$, $B \in \text{co } \tau'_2$) there are disjoint sets $U \in \tau'_j$, $V \in \tau_i$ ($U \in \tau_2$, $V \in \tau_1$) such that $A \subset U$ and $B \subset V$.

Evidently, if (X, τ_1, τ_2) is p -normal, then for every subset $Y \in \text{co } \tau_i$ the BsS (Y, τ'_1, τ'_2) is (i, j) -WS-supernormal in X and so, if (X, τ_1, τ_2) is p -normal and $Y \in \text{co } \tau_1 \cap \text{co } \tau_2$, then (Y, τ'_1, τ'_2) is p -WS-supernormal in X (clearly, for a BS $(X, \tau_1 < \tau_2)$ the last fact is correct for $Y \in \text{co } \tau_1$).

Example 2.17. Let $(\mathbb{R}, \omega_1, \omega_2)$ be the natural BS, that is, $\omega_1 = \{\emptyset, \mathbb{R}\} \cup \{(a, +\infty) : a \in \mathbb{R}\}$ and $\omega_2 = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, a) : a \in \mathbb{R}\}$ are the lower and upper topologies, respectively. Then it is not difficult to see that every set $Y \in \omega_i \setminus \{\emptyset\}$ is (i, j) -WS-supernormal in the p -normal BS $(\mathbb{R}, \omega_1, \omega_2)$, but it is not (j, i) -WS-supernormal in $(\mathbb{R}, \omega_1, \omega_2)$. Moreover, if $Y = [a, b] = (-\infty, b] \cap [a, +\infty) \in p\text{-Cl}(\mathbb{R})$, then it is p -WS-supernormal in $(\mathbb{R}, \omega_1, \omega_2)$.

Proposition 2.18. In a BS $(X, \tau_1 < \tau_2)$ we have:

(1) If $Y \in \tau_2$ and $\tau_1 <_N \tau_2$, then the following implications hold:

$$\begin{array}{ccc} (Y, \tau'_1, \tau'_2) \text{ is 2-WS-supernorm. in } X & \implies & (Y, \tau'_1, \tau'_2) \text{ is (1, 2)-WS-supernorm. in } X \\ \Downarrow & & \Downarrow \\ (Y, \tau'_1, \tau'_2) \text{ is (2, 1)-WS-supernorm. in } X & \implies & (Y, \tau'_1, \tau'_2) \text{ is 1-WS-supernorm. in } X. \end{array}$$

(2) If $\tau_1 <_C \tau_2$, then

$$(Y, \tau'_1, \tau'_2) \text{ is } p\text{-strongly normal in } X \implies (Y, \tau'_1, \tau'_2) \text{ is 1-strongly normal in } X$$

and if $\tau_1 <_N \tau_2$, then

$$\begin{array}{ccc} Y \text{ is 2-strongly normal in } X & \implies & Y \text{ is } d\text{-strongly normal in } X \\ \Downarrow & & \Downarrow \\ Y \text{ is } p\text{-strongly normal in } X & \implies & Y \text{ is 1-strongly normal in } X \end{array}$$

Proof. (1) First of all, let us note that if $Y \in \tau_2$, then by (2) of Corollary 2.3.13 in [15], $\tau_1 <_N \tau_2$ implies $\tau'_1 <_N \tau'_2$.

For the upper horizontal implication let $A \in \text{co } \tau'_1$, $B \in \text{co } \tau'_2$ and $A \cap B = \emptyset$. Since $A \in \text{co } \tau'_1 \subset \text{co } \tau'_2$ and (Y, τ'_1, τ'_2) is 2-WS-supernormal in X , there are neighborhoods $U'(A) \in \tau'_2 \subset \tau_2$, $U'(B) \in \tau_2$ such that $U'(A) \cap U'(B) = \emptyset$ and so $\tau_2 \text{ cl } U'(A) \cap U'(B) = \emptyset$. According to (2) of Corollary 2.3.12 in [15],

$$U'(A) \subset \tau_2 \text{ int } \tau_2 \text{ cl } U'(A) = \tau_1 \text{ int } \tau_2 \text{ cl } U'(A) = U''(A) \in \tau_1 \subset \tau_2$$

and for the set $U(A) = U''(A) \cap Y \in \tau'_1 \subset \tau'_2$ we have $U(A) \cap \tau_1 \text{ cl } U'(B) = \emptyset$. Moreover, by Corollary 2.3.10 in [15],

$$\tau_1 <_N \tau_2 \implies \tau_1 <_C \tau_2$$

and (2) of Corollary 2.2.7 in [15] gives that

$$U'(B) \subset \tau_2 \text{ int } \tau_1 \text{ cl } U'(B) = \tau_1 \text{ int } \tau_1 \text{ cl } U'(B) = U(B) \in \tau_1.$$

Now, it is clear that $U(A) \cap U(B) = \emptyset$ and, hence, (Y, τ'_1, τ'_2) is $(2, 1)$ -WS-supernormal in X .

For the lower horizontal implication let $A \in \text{co } \tau'_1$, $B \in \text{co } \tau_1$ and $A \cap B = \emptyset$. Since $A \in \text{co } \tau'_1 \subset \text{co } \tau'_2$ and (Y, τ'_1, τ'_2) is $(2, 1)$ -WS-supernormal in X , there are neighborhoods $U(A) \in \tau'_1$, $U'(B) \in \tau_2$ such that $U(A) \cap U'(B) = \emptyset$. Since $U(A) \in \tau'_1 \subset \tau'_2 \subset \tau_2$, we have $U(A) \cap \tau_2 \text{ cl } U'(B) = \emptyset$. But

$$U'(B) \subset \tau_2 \text{ int } \tau_2 \text{ cl } U'(B) = \tau_1 \text{ int } \tau_2 \text{ cl } U'(B) = U(B) \in \tau_1$$

and $U(A) \cap U(B) = \emptyset$. Thus (Y, τ'_1, τ'_2) is 1-WS-supernormal in X .

For the left vertical implication let $A \in \text{co } \tau'_2$, $B \in \text{co } \tau_1$ and $A \cap B = \emptyset$. Since $B \in \text{co } \tau_1 \subset \text{co } \tau_2$ and (Y, τ'_1, τ'_2) is 2-WS-supernormal in X , there are neighborhoods $U'(A) \in \tau'_2$, $U(B) \in \tau_2$ such that $U'(A) \cap U(B) = \emptyset$ and so $\tau_2 \text{ cl } U'(A) \cap U(B) = \emptyset$. Furthermore, according to (2) of Corollary 2.3.12 in [15],

$$U'(A) \subset \tau'_2 \text{ int } \tau'_2 \text{ cl } U'(A) = \tau'_1 \text{ int } \tau'_2 \text{ cl } U'(A) = U(A) \in \tau'_1$$

and $U(A) \cap U(B) = \emptyset$ so that (Y, τ'_1, τ'_2) is $(2, 1)$ -WS-supernormal in X .

Finally, for the right vertical implication let $A \in \text{co } \tau'_1$, $B \in \text{co } \tau_1$ and $A \cap B = \emptyset$. Since $B \in \text{co } \tau_1 \subset \text{co } \tau_2$ and (Y, τ'_1, τ'_2) is $(1, 2)$ -WS-supernormal in X , there are neighborhoods $U'(A) \in \tau'_2$, $U(B) \in \tau_1$ such that $U'(A) \cap U(B) = \emptyset$. Hence $\tau_1 \text{ cl } U'(A) \cap U(B) = \emptyset$ and by (2) of Corollary 2.3.12 in [15],

$$U'(A) \subset \tau'_2 \text{ int } \tau'_2 \text{ cl } U'(A) = \tau'_1 \text{ int } \tau'_2 \text{ cl } U'(A) = U(A) \in \tau'_1.$$

Clearly $U(A) \cap U(B) = \emptyset$ and, thus, (Y, τ'_1, τ'_2) is 1-WS-supernormal in X .

(2) First, let $A, B \in \text{co } \tau'_1$ and $A \cap B = \emptyset$. Since $B \in \text{co } \tau'_1 \subset \text{co } \tau'_2$ and (Y, τ'_1, τ'_2) is p -strongly normal in X , there are neighborhoods $U'(A) \in \tau_2$, $U(B) \in \tau_1$ such that $U'(A) \cap U(B) = \emptyset$. Clearly, $\tau_1 \text{ cl } U'(A) \cap U(B) = \emptyset$ and according to (2) of Corollary 2.2.7 in [15],

$$U'(A) \subset \tau_2 \text{ int } \tau_1 \text{ cl } U'(A) = \tau_1 \text{ int } \tau_1 \text{ cl } U'(A) = U(A) \in \tau_1,$$

where $U(A) \cap U(B) = \emptyset$. Thus (Y, τ'_1, τ'_2) is 1-strongly normal in X .

Furthermore, let $A \in \text{co } \tau'_1$, $B \in \text{co } \tau'_2$ and $A \cap B = \emptyset$. Since $A \in \text{co } \tau'_1 \subset \text{co } \tau'_2$ and (Y, τ'_1, τ'_2) is 2-strongly normal in X , there are neighborhoods $U(A)$, $U'(B) \in \tau_2$ such that $U(A) \cap U'(B) = \emptyset$ and so $U(A) \cap \tau_2 \text{ cl } U'(B) = \emptyset$. Hence, by (2) of Corollary 2.3.12 in [15],

$$U'(B) \subset \tau_2 \text{ int } \tau_2 \text{ cl } U'(B) = \tau_1 \text{ int } \tau_2 \text{ cl } U'(B) = U(B) \in \tau_1, \quad U(A) \cap U(B) = \emptyset$$

and, hence, (Y, τ'_1, τ'_2) is p -strongly normal in X . The horizontal implications follow from the first implication of (2), since $\tau_1 <_N \tau_2$ implies $\tau_1 <_C \tau_2$. \square

Below we shall study the interrelation of the notions of relative WS-supernormality and relative strong normality for both the topological and the bitopological cases.

Proposition 2.19. *For a BsS (Y, τ'_1, τ'_2) of a BS (X, τ_1, τ_2) the following conditions are satisfied:*

(1) If $Y \in \text{co } \tau_i$ and (Y, τ'_1, τ'_2) is p -strongly normal in X , then

$$(Y, \tau'_1, \tau'_2) \text{ is } (i, j)\text{-WS-supernormal in } X.$$

(2) If $Y \in \tau_j$ and (Y, τ'_1, τ'_2) is (i, j) -WS-supernormal in X , then

$$(Y, \tau'_1, \tau'_2) \text{ is } p\text{-strongly normal in } X.$$

Proof. (1) Let $Y \in \text{co } \tau_i$, $A \in \text{co } \tau'_i$, $B \in \text{co } \tau_j$ and $A \cap B = \emptyset$. Then $B' = B \cap Y \in \text{co } \tau'_j$ and since (Y, τ'_1, τ'_2) is p -strongly normal in X , there are disjoint sets $U' \in \tau_j$, $V' \in \tau_i$ such that $A \subset U'$ and $B' \subset V'$. Evidently, if $U = U' \cap Y$, $V = V' \cup (X \setminus Y)$, then $U \in \tau'_j$, $V \in \tau_i$, $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$ so that (Y, τ'_1, τ'_2) is (i, j) -WS-supernormal in X .

(2) Let $Y \in \tau_j$, $A \in \text{co } \tau'_i$, $B \in \text{co } \tau'_j$ and $A \cap B = \emptyset$. Then $A \cap \tau_j \text{ cl } B = \emptyset$ and since (Y, τ'_1, τ'_2) is (i, j) -WS-supernormal in X , there are sets $U \in \tau'_j$, $V \in \tau_i$ such that $A \subset U$, $\tau_j \text{ cl } B \subset V$ and $U \cap V = \emptyset$. Clearly $U \in \tau_j$, $B \subset V$ and, hence, (Y, τ'_1, τ'_2) is p -strongly normal in X . \square

Corollary 2.20. For a BsS (Y, τ'_1, τ'_2) of a BS (X, τ_1, τ_2) the following conditions are satisfied:

(1) If $Y \in \text{co } \tau_1 \cap \text{co } \tau_2$ and (Y, τ'_1, τ'_2) is p -strongly normal in X , then

$$(Y, \tau'_1, \tau'_2) \text{ is } p\text{-WS-supernormal in } X.$$

(2) If $Y \in \tau_1 \cap \tau_2$ and (Y, τ'_1, τ'_2) is p -WS-supernormal in X , then

$$(Y, \tau'_1, \tau'_2) \text{ is } p\text{-strongly normal in } X.$$

(3) If $Y \in \tau_1 \cap \tau_2 \cap \text{co } \tau_1 \cap \text{co } \tau_2$, then the following conditions are equivalent: (Y, τ'_1, τ'_2) is $(1, 2)$ -WS-supernormal in X , (Y, τ'_1, τ'_2) is $(2, 1)$ -WS-supernormal in X , (Y, τ'_1, τ'_2) is p -WS-supernormal in X and (Y, τ'_1, τ'_2) is p -strongly normal in X .

Therefore, the equivalences remain valid for a BS $(X, \tau_1 < \tau_2)$ if $Y \in \tau_1 \cap \text{co } \tau_1$.

Proof. (1) and (2) follow directly from (1) and (2) of Proposition 2.19, respectively.

(3) By (1) and (2) of Proposition 2.19, if $Y \in \tau_2 \cap \text{co } \tau_1$, then (Y, τ'_1, τ'_2) is $(1, 2)$ -WS-supernormal in X if and only if (Y, τ'_1, τ'_2) is p -strongly normal in X and if $Y \in \tau_1 \cap \text{co } \tau_2$, then (Y, τ'_1, τ'_2) is $(2, 1)$ -WS-supernormal in X if and only if (Y, τ'_1, τ'_2) is p -strongly normal in X .

The rest is obvious. \square

Corollary 2.21. For a TsS (Y, τ') of a TS (X, τ) the following conditions are satisfied:

(1) If $Y \in \text{co } \tau$ and (Y, τ') is strongly normal in X , then (Y, τ') is WS-supernormal in X and if $Y \in \tau$ and (Y, τ') is WS-supernormal in X , then (Y, τ') is strongly normal in X .

- (2) If $Y \in \tau \cap \text{co } \tau$, then (Y, τ') is WS-supernormal in X if and only if (Y, τ') is strongly normal in X .

In connection with Corollary 2.21 and so, with the bitopological case, we consider it necessary to give the following elementary

Example 2.22. Let $X = \{a, b, c, d, e\}$, $\tau = \{\emptyset, \{c\}, \{a, b, c\}, \{c, d, e\}, X\}$ and $Y \in \{a, b, d\}$. Then (Y, τ') is WS-supernormal in X , but it is not strongly normal in X .

Now, if $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{d, e\}, \{a, d, e\}, \{b, d, e\}, \{a, b, d, e\}, X\}$ and $Y = \{a, d, e\}$, then (Y, τ') is strongly normal in X , but it is not WS-supernormal in X .

Our next theorem, which is one of the main results of this work, concerns the improvement of Theorem 3.2.26 and its Corollaries 3.2.27–3.2.30 in [15].

Theorem 2.23. If a d -second countable and p -normal BS (X, τ_1, τ_2) can be represented as a union of a sequence F_1, F_2, \dots of p -closed sets, where F_k is p -WS-supernormal in X and $p\text{-ind } F_k = 0$ for each $k = \overline{1, \infty}$, then $p\text{-Ind } X = 0$.

Proof. Let $A \in \text{co } \tau_1$, $B \in \text{co } \tau_2$ and $A \cap B = \emptyset$. We shall prove that there exist sets $G \in \tau_2$, $H \in \tau_1$ such that

$$A \subset G, \quad B \subset H, \quad G \cap H = \emptyset \quad \text{and} \quad G \cup H = X \quad (1)$$

and so the empty set is a partition, corresponding to the pair (A, B) .

Since $A \in \text{co } \tau_1$, $B \in \text{co } \tau_2$ and $A \cap B = \emptyset$, by Corollary 0.1.8 in [15], there exist sets $U_0 \in \tau_2$, $V_0 \in \tau_1$ such that

$$A \subset U_0, \quad B \subset V_0 \quad \text{and} \quad \tau_1 \text{ cl } U_0 \cap \tau_2 \text{ cl } V_0 = \emptyset. \quad (2)$$

We shall define inductively two sequences of 2-open and 1-open sets U_0, U_1, \dots and V_0, V_1, \dots , respectively, satisfying for each $k = \overline{1, \infty}$ the following conditions:

$$U_{k-1} \subset U_k, \quad V_{k-1} \subset V_k \quad \text{if } k \geq 1, \quad \text{and} \quad \tau_1 \text{ cl } U_k \cap \tau_2 \text{ cl } V_k = \emptyset \quad (3)$$

$$F_k \subset U_k \cup V_k, \quad \text{where } F_0 = \emptyset. \quad (4)$$

Clearly, the sets U_0 and V_0 , defined above, satisfy all conditions for $k = 0$. Assume that the sets U_k and V_k , satisfying (3) and (4), are defined for $k < p$. If $F_p = \tau_1 \text{ cl } F_p \cap \tau_2 \text{ cl } F_p$, then the sets $\tau_1 \text{ cl } U_{p-1} \cap F_p \in \text{co } \tau'_1$ and $\tau_2 \text{ cl } V_{p-1} \cap F_p \in \text{co } \tau'_2$ are disjoint in (F_p, τ'_1, τ'_2) . Since $p\text{-ind } F_p = 0$, according to Theorem 3.2.12 in [15], the empty set is a partition between $\tau_1 \text{ cl } U_{p-1} \cap F_p$ and $\tau_2 \text{ cl } V_{p-1} \cap F_p$ in F_p . Hence, there exists a set $V \in \tau'_2 \cap \text{co } \tau'_1$ such that

$$\tau_1 \text{ cl } U_{p-1} \cap F_p \subset V \quad \text{and} \quad \tau_2 \text{ cl } V_{p-1} \cap F_p \subset F_p \setminus V.$$

It is clear that $\tau_1 \text{ cl } V \cap \tau_2 \text{ cl } (F_p \setminus V) = \emptyset$ as $\tau_1 \text{ cl } V \subset \tau_1 \text{ cl } F_p$, $\tau_2 \text{ cl } (F_p \setminus V) \subset \tau_2 \text{ cl } F_p$ and $\tau_1 \text{ cl } F_p \cap \tau_2 \text{ cl } F_p = F_p$. Moreover, $V \in \text{co } \tau'_1$, $\tau_2 \text{ cl } V_{p-1} \in \text{co } \tau_2$ and $V \cap \tau_2 \text{ cl } V_{p-1} = \emptyset$. Since (F_p, τ'_1, τ'_2) is $(1, 2)$ -WS-supernormal in X , there are disjoint sets $U \in \tau'_2$, $W \in \tau_1$ such that $V \subset U$ and $\tau_2 \text{ cl } V_{p-1} \subset W$. Since $V \cap W = \emptyset$ and $W \in \tau_1$, we have $\tau_1 \text{ cl } V \cap W = \emptyset$ so that $\tau_1 \text{ cl } V \cap \tau_2 \text{ cl } V_{p-1} = \emptyset$.

By the similar manner, taking into account that (F_p, τ'_1, τ'_2) is $(2, 1)$ -WS-supernormal in X , one can prove that $\tau_2 \text{ cl}(F_p \setminus V) \cap \tau_1 \text{ cl} U_{p-1} = \emptyset$.

Let $\Phi_1 = \tau_1 \text{ cl} U_{p-1} \cup \tau_1 \text{ cl} V$ and $\Phi_2 = \tau_2 \text{ cl} V_{p-1} \cup \tau_2 \text{ cl}(F_p \setminus V)$. Then $\Phi_i \in \text{co } \tau_i$ and $\Phi_1 \cap \Phi_2 = \emptyset$. Since (X, τ_1, τ_2) is p -normal, there exist sets $U_p \in \tau_2$, $V_p \in \tau_1$ such that

$$\begin{aligned} U_{p-1} \subset \Phi_1 \subset U_p, \quad V_{p-1} \subset \Phi_2 \subset V_p, \\ \tau_1 \text{ cl} U_p \cap \tau_2 \text{ cl} V_p = \emptyset \quad \text{and} \quad F_p \subset U_p \cup V_p \end{aligned}$$

so that the sets $\{U_p\}_{p=1}^\infty$ and $\{V_p\}_{p=1}^\infty$ satisfy (3) and (4) for $k = p$. Thus the construction of the sequences U_0, U_1, \dots and V_0, V_1, \dots is completed. It follows from (2), (3) and (4) that the unions $G = \bigcup_{p=1}^\infty U_p$ and $H = \bigcup_{p=1}^\infty V_p$ satisfy (1). \square

Corollary 2.24. *If a p -normal BS (X, τ_1, τ_2) can be represented as a union of a sequence F_1, F_2, \dots of p -closed sets, where F_k is p -WS-supernormal in X and $p\text{-Ind } F_k = 0$ for each $k = \overline{1, \infty}$, then $p\text{-Ind } X = 0$.*

The proof of this corollary is analogous to that of Theorem 2.23, the only difference being that since $p\text{-Ind } F_k = 0$ for each $k = \overline{1, \infty}$, Theorem 3.2.12 from [15] is unnecessary and so is the requirement of d -second countability.

Note further that if $F_k \in \text{co } \tau_1 \cap \text{co } \tau_2$, then the requirement of p -WS-supernormality of each F_k in X is also unnecessary since it is automatically satisfied by the reasoning before Example 2.17 provided that (X, τ_1, τ_2) is p -normal. Hence, in this case we obtain Corollary 3.2.25 from [15].

Corollary 2.25. *If a (d -second countable) p -normal BS $(X, \tau_1 < \tau_2)$ can be represented as a union of a sequence F_1, F_2, \dots of 2-closed sets, where F_k is $(1, 2)$ -WS-supernormal in X and $(p\text{-ind } F_k = 0)$ $p\text{-Ind } F_k = 0$ for each $k = \overline{1, \infty}$, then $p\text{-Ind } X = 0$.*

Proof. On the one hand, $F_k \in \text{co } \tau_2 \subset p\text{-Cl}(X)$ and on the other hand, $F_k \in \text{co } \tau_2$ implies that F_k is $(2, 1)$ -WS-supernormal in X . Hence, the case of brackets follows directly from Theorem 2.23, and the case without brackets – from Corollary 2.24. \square

Corollary 2.26. *If a (d -second countable) p -normal BS (X, τ_1, τ_2) can be represented as a union of a sequence F_1, F_2, \dots , where every F_k is a countable union of p -closed sets, that is, $F_k = \bigcup_{n=1}^\infty F_n^k$, $(p\text{-ind } F_k = 0)$ $p\text{-Ind } F_k = 0$ for each $k = \overline{1, \infty}$ and F_n^k is p -WS-supernormal in X for each $n = \overline{1, \infty}$, $k = \overline{1, \infty}$, then $p\text{-Ind } X = 0$. Moreover, if (X, τ_1, τ_2) is $R\text{-}p\text{-T}_1$, then $p\text{-ind } X = 0$.*

Proof. For the case of brackets, note that by (2) of Corollary 3.1.5 in [15], $p\text{-ind } F_n^k = 0$ and for the case without brackets, by Corollary 3.2.8 in [15], $p\text{-Ind } F_n^k = 0$ for each $n = \overline{1, \infty}$, $k = \overline{1, \infty}$. Therefore, following Theorem 2.23 and Corollary 2.24, respectively, $p\text{-Ind } X = 0$, since $X = \bigcup_{k=1}^\infty \bigcup_{n=1}^\infty F_n^k$.

In the case where (X, τ_1, τ_2) is $R\text{-}p\text{-T}_1$, it suffices to use Corollary 3.2.13 from [15]. \square

Corollary 2.27. *If a $R\text{-}p\text{-T}_1$, d -second countable and p -normal BS*

$(X, \tau_1 < \tau_2)$ can be represented as a union of two BS's Y and Z , where $p\text{-Ind } Y = p\text{-Ind } Z = 0$ and one of them is 1-open, then $p\text{-Ind } X = 0$.

Proof. Let, for example, $Y \in \tau_1$. Then $X \setminus Y \in \text{co } \tau_1 \subset p\text{-Cl}(X)$ and since $X \setminus Y \subset Z$, where $p\text{-Ind } Z = 0$, by Corollary 3.2.8 in [15], $p\text{-Ind}(X \setminus Y) = 0$. Furthermore, by Corollary 0.1.13 in [15], the BS (X, τ_1, τ_2) is p -perfectly normal and, hence, $Y \in \tau_1 \subset \tau_2 \subset 1\mathcal{F}_\sigma(X)$, that is, $Y = \bigcup_{k=1}^{\infty} F_k$, where $F_k \in \text{co } \tau_1 \subset p\text{-Cl}(X)$ for each $k = \overline{1, \infty}$. Thus, once more applying Corollary 3.2.8 from [15], we obtain that $p\text{-Ind } F_k = 0$ for each $k = \overline{1, \infty}$, since $p\text{-Ind } Y = 0$. Evidently, $X = Y \cup (X \setminus Y) = \bigcup_{k=1}^{\infty} F_k \cup (X \setminus Y)$ and according to Corollary 2.24, $p\text{-Ind } X = 0$, since $X \setminus Y \in \text{co } \tau_1$ and $F_k \in \text{co } \tau_1$ for each $k = \overline{1, \infty}$ imply that each of them is p -WS-supernormal in X . \square

Below, in addition to Theorems 3.1.36 and 3.2.28 from [15], we have

Proposition 2.28. *The following conditions are satisfied for a BS (X, τ_1, τ_2) :*

- (1) *If $\tau_1 C \tau_2$ and (X, τ_1, τ_2) is $(2, 1)$ -regular (or $(2, 1)\text{-ind}(X)$ is finite), then $(2, 1)\text{-ind } X \leq 2\text{-ind } X$.*
- (2) *If $\tau_1 N \tau_2$ and (X, τ_1, τ_2) is 2-regular (or $2\text{-ind}(X)$ is finite), then $1\text{-ind } X \leq (1, 2)\text{-ind } X$.*
- (3) *If $\tau_1 <_C \tau_2$ and (X, τ_1, τ_2) is $(2, 1)$ -regular or $\tau_1 <_N \tau_2$ and (X, τ_1, τ_2) is 2-regular, then*

$$1\text{-ind } X = (1, 2)\text{-ind } X = 2\text{-ind } X = (2, 1)\text{-ind } X.$$

- (4) *If $\tau_1 <_C \tau_2$, (X, τ_1, τ_2) is 1- T_1 and p -normal (or $(i, j)\text{-Ind } X$ is finite), then*

$$1\text{-Ind } X = (1, 2)\text{-Ind } X = 2\text{-Ind } X = (2, 1)\text{-Ind } X.$$

Proof. (1) If $(X, \tau_1 C \tau_2)$ is $(2, 1)$ -regular, then by Corollary 2.2.9 in [15], $\tau_2 \subset \tau_1$ and it remains to use the first inequality in (1) of Theorem 3.1.36 in [15].

(2) If $(X, \tau_1 N \tau_2)$ is 2-regular, then by Theorem 1 in [22], $\tau_2 \subset \tau_1$ and it remains to use the second inequality in (1) of Theorem 3.1.36 in [15].

(3) If $(X, \tau_1 <_C \tau_2)$ is 2-regular, then by (1) or by (2) above, $\tau_1 = \tau_2$.

(4) If $(X, \tau_1 <_C \tau_2)$ is 1- T_1 and p -normal, then by the implications before Definition 0.1.7 in [15], (X, τ_1, τ_2) is $(2, 1)$ -regular and so, by (1) above, $\tau_1 = \tau_2$.

Finally, note that if $(2, 1)\text{-ind } X$ ($2\text{-ind } X$, $(i, j)\text{-Ind } X$) is finite, then by (1) of Proposition 3.1.4 in [15] (the well-know topological fact, (1) of Proposition 3.2.7 in [15]), the BS (X, τ_1, τ_2) is $(2, 1)$ -regular (2-regular, p -normal). \square

At the end of this section we consider the notion of almost n -dimensionality [2] from the bitopological point of view.

Definition 2.29. We say that a TS (X, τ_2) is almost n -dimensional in the sense of the small (large) inductive dimension and we write $\text{aInd}(X, \tau_2) = n$ ($\text{Ind}(X, \tau_2) = n$) if there exists a topology τ_1 on X such that $\text{ind}(X, \tau_1) \leq n$ ($\text{Ind}(X, \tau_1) \leq n$),

τ_1 is weaker than the given topology τ_2 on X , the BS $(X, \tau_1 < \tau_2)$ is $(2,1)$ -regular (p -normal) and n is the smallest natural number such that a topology τ_1 exists for n .

In this case we also say that the TS (X, τ_2) is almost n -dimensional (in both senses) owing to the (weaker) topology τ_1 on X .

Theorem 2.30. *For a TS (X, τ_2) we have:*

- (1) *If $\text{aInd}(X, \tau_2) = n$ owing to τ_1 , then $(1,2)\text{-ind}(X, \tau_1, \tau_2) \leq n$, the BS (X, τ_1, τ_2) is d -regular and p -regular, and τ_1 is a cotopology of τ_2 in the sense of [1].*
- (2) *If $\text{aInd}(X, \tau_2) = n$ owing to τ_1 , then (X, τ_1, τ_2) is 1-normal; moreover, if (X, τ_1, τ_2) is 1- T_1 , then $\text{aInd}(X, \tau_2) \leq n$.*
- (3) *If $\text{aInd}(X, \tau_2) = n$ owing to τ_1 , then for each topology τ on X such that $\tau_1 \subset \tau \subset \tau_2$, the BS (X, τ, τ_2) is $(2,1)$ -regular, 2-regular and $(1,2)\text{-ind}(X, \tau_1, \tau) \leq n$, the BS (X, τ_1, τ) is 1-regular and $(1,2)$ -regular.*
- (4) *If $\text{aInd}(X, \tau_2) = n$ owing to τ_1 , (X, τ_1, τ_2) is 1- T_2 and 2-locally compact, then (X, τ_1, τ_2) is a 2-Baire space (and, hence, it is also an almost $(2,1)$ -Baire space, a 2-weak Baire space and a $(2,1)$ -weak Baire space in the sense of [15]); moreover, if, in addition, (X, τ_1, τ_2) is 1-second countable, then the TS (X, τ_2) has a bounded complete computational model in the sense of [8].*

Proof. (1) By Definition 2.29 and (1) of Theorem 3.1.36 in [15],

$$(1,2)\text{-ind}(X, \tau_1, \tau_2) \leq 1\text{-ind}(X, \tau_1, \tau_2) \leq n,$$

where the right inequality gives that (X, τ_1, τ_2) is 1-regular. But it is also $(2,1)$ -regular and, hence, it is d -regular and p -regular as $\tau_1 \subset \tau_2$. The fact that τ_1 is a cotopology of τ_2 follows directly from Definition 2.29 and Theorem 7.3.20 in [15].

(2) Since $1\text{-Ind}(X, \tau_1, \tau_2) \leq n$, (X, τ_1, τ_2) is 1-normal. If (X, τ_1, τ_2) is 1- T_1 ($\iff R\text{-}p\text{-}T_1$), then (X, τ_1, τ_2) is p -normal implies that (X, τ_1, τ_2) is p -regular and so, $(2,1)$ -regular. Moreover,

$$1\text{-ind}(X, \tau_1, \tau_2) \leq 1\text{-Ind}(X, \tau_1, \tau_2) \leq n,$$

i.e., $\text{ind}(X, \tau_1) \leq \text{Ind}(X, \tau_1)$ and by Definition 2.29, $\text{aInd}(X, \tau_2) \leq n$.

(3) Follows directly from (1) above and (1) of Theorem 3.1.36 in [15] as $\tau_1 \subset \tau_2$.

(4) The first part follows directly from (1) and (2) of Corollary 7.3.25 in [15]. The second part is an immediate consequence of (1) and (2) of Corollary 7.3.25 in [15] taking into account (3) of Theorem 9 in [8], since, by (1) above, (X, τ_1, τ_2) is p -regular. \square

3. (i, j) -Submaximal, (i, j) -Nodec, (i, j) - \mathcal{I} and (i, j) - D -Spaces

Definition 3.1. A BS (X, τ_1, τ_2) is said to be (i, j) -submaximal if every subset of X is (i, j) -locally closed, that is, if $2^X = (i, j)\text{-}\mathcal{LC}(X)$.

Taking into account the inclusions after Definition 1.3, for a BS $(X, \tau_1 < \tau_2)$ the following implications hold:

$$\begin{array}{ccc} (X, \tau_1, \tau_2) \text{ is 1-submaximal} & \implies & (X, \tau_1, \tau_2) \text{ is } (2, 1)\text{-submaximal} \\ \Downarrow & & \Downarrow \\ (X, \tau_1, \tau_2) \text{ is } (1, 2)\text{-submaximal} & \implies & (X, \tau_1, \tau_2) \text{ is 2-submaximal.} \end{array}$$

Therefore, from Corollary 1.3 in [5] it follows immediately that if $(X, \tau_1 < \tau_2)$ is i -submaximal or (i, j) -submaximal, then it is 2-nodc.

Theorem 3.2. *For a BS (X, τ_1, τ_2) the following conditions are equivalent:*

- (1) (X, τ_1, τ_2) is (i, j) -submaximal.
- (2) Every subset of X is $\text{co}(i, j)$ -locally closed.
- (3) Every j -boundary subset of X is i -closed.
- (4) $\tau_j \text{ cl } A \setminus A$ is i -closed for every subset $A \subset X$.
- (5) Every j -dense subset of X is i -open.

Proof. It is clear that $(1) \iff (2)$ since $2^X = (i, j)\text{-}\mathcal{LC}(X)$ implies the equivalence $A \in (i, j)\text{-}\mathcal{LC}(X) \iff X \setminus A \in (i, j)\text{-}\mathcal{LC}(X)$ so that $A \in \text{co}(i, j)\text{-}\mathcal{LC}(X)$ for each subset $A \subset X$.

$(4) \iff (5)$ If $A \in j\text{-}\mathcal{D}(X)$, then, by (4), $\tau_j \text{ cl } A \setminus A = X \setminus A \in \text{co } \tau_i$ so that $A \in \tau_i$. Conversely, if there is a set $A \subset X$ such that $(\tau_j \text{ cl } A \setminus A) \not\subseteq \text{co } \tau_i$, then

$$X \setminus (\tau_j \text{ cl } A \setminus A) = \tau_j \text{ int}(X \setminus A) \cup A \not\subseteq \tau_i.$$

Hence, by (5), $\tau_j \text{ int}(X \setminus A) \cup A \not\subseteq j\text{-}\mathcal{D}(X)$.

$(4) \implies (1)$ Let $A \subset X$ be any set. Then, by (4), $\tau_j \text{ cl } A \setminus A = \tau_j \text{ cl } A \cap (X \setminus A) \in \text{co } \tau_i$ so that

$$X \setminus (\tau_j \text{ cl } A \setminus A) = (X \setminus \tau_j \text{ cl } A) \cup A \in \tau_i.$$

But

$$A = \tau_j \text{ cl } A \cap ((X \setminus \tau_j \text{ cl } A) \cup A) = F \cap U,$$

where $F \in \text{co } \tau_j$ and $U \in \tau_i$; hence $A \in (i, j)\text{-}\mathcal{LC}(X)$.

$(2) \implies (3)$ If $\tau_j \text{ int } A = \emptyset$, then the equivalence $A \in \text{co}(i, j)\text{-}\mathcal{LC}(X) \iff (A = U \cup F, \text{ where } U \in \tau_j \text{ and } F \in \text{co } \tau_i)$ implies that $U = \emptyset$ and, thus, $A \in \text{co } \tau_i$.

The implication $(3) \implies (4)$ is obvious, since $\tau_j \text{ cl } A \setminus A \in j\text{-}\mathcal{Bd}(X)$ for each subset $A \subset X$. \square

Note also here that according to (3) of Theorem 3.2 above and Corollary 1.3 in [5], for a BS $(X, \tau_1 S \tau_2)$ we have:

$$\begin{aligned} (X, \tau_1, \tau_2) \text{ is } (1, 2)\text{-submaximal} & \iff (X, \tau_1, \tau_2) \text{ is 1-submaximal} \implies \\ & \implies (X, \tau_1, \tau_2) \text{ is 1-nodc} \end{aligned}$$

and

$$(X, \tau_1, \tau_2) \text{ is } (2, 1)\text{-submaximal} \iff (X, \tau_1, \tau_2) \text{ is 2-submaximal} \implies$$

$$\implies (X, \tau_1, \tau_2) \text{ is 2-nodex,}$$

since, by (2) of Theorem 2.1.5 in [15], if $\tau_1 S \tau_2$, then $1\text{-}\mathcal{Bd}(X) = 2\text{-}\mathcal{Bd}(X)$.

The other relations between submaximal and nodex spaces as well as between their bitopological modifications will be given in Corollary 3.7.

Corollary 3.3. *For a $j\text{-T}_1$ BS (X, τ_1, τ_2) the following conditions are equivalent:*

- (1) *Every j -boundary subset of X is i -discrete.*
- (2) *$\tau_j \text{ cl } A \setminus A$ is i -discrete for every subset $A \subset X$,
and in the case where (X, τ_1, τ_2) is (i, j) -submaximal, each of them is satisfied.*

Moreover, if $X_j^i = \emptyset$, then (1) (and so (2)) is equivalent to the (i, j) -submaximality of (X, τ_1, τ_2) .

Proof. It is clear that for any subset $A \subset X$ the set $\tau_j \text{ cl } A \setminus A \in j\text{-}\mathcal{Bd}(X)$ and, hence, by (1), $\tau_j \text{ cl } A \setminus A = (\tau_j \text{ cl } A \setminus A)_i^i$. Conversely, if $A \in j\text{-}\mathcal{Bd}(X)$ (i.e., if $X \setminus A \in j\text{-}\mathcal{D}(X)$), then by (2),

$$\tau_j \text{ cl}(X \setminus A) \setminus (X \setminus A) = (\tau_j \text{ cl}(X \setminus A) \setminus (X \setminus A))_i^i,$$

that is, $X \setminus (X \setminus A) = A = A_i^i$.

Now, let (X, τ_1, τ_2) be (i, j) -submaximal and let us prove that (3) of Theorem 3.2 implies (1) of this corollary. Indeed: if there is a set $A \in j\text{-}\mathcal{Bd}(X)$ such that $A \neq A_i^i$, i.e., $A \setminus A_i^i \neq \emptyset$, then there exists a point $x \in A$ such that for each i -open neighborhood $U(x)$: $U(x) \cap (A \setminus \{x\}) \neq \emptyset$. If $B = A \setminus \{x\}$, then $x \in \tau_i \text{ cl } B$ so that $B \neq \tau_i \text{ cl } B$. But $B \subset A$ implies that $B \in j\text{-}\mathcal{Bd}(X)$. A contradiction with (3) of Theorem 3.2.

Finally, let us prove that if $X_j^i = \emptyset$, then (1) of this corollary implies (3) of Theorem 3.2.

Let $A \subset X$, $A \in j\text{-}\mathcal{Bd}(X)$ and $A \neq \tau_i \text{ cl } A$. If $x \in \tau_i \text{ cl } A \setminus A$ is any point, then for each i -open neighborhood $U(x)$ we have $U(x) \cap (A \setminus \{x\}) = U(x) \cap A \neq \emptyset$. Let $B = A \cup \{x\}$, where $\{x\} \in \text{co } \tau_j$ as (X, τ_1, τ_2) is $j\text{-T}_1$. Then, by the well-known topological fact,

$$\tau_j \text{ int } B = \tau_j \text{ int}(A \cup \{x\}) = \tau_j \text{ int}(\tau_j \text{ int } A \cup \{x\}) = \tau_j \text{ int}\{x\} = \emptyset$$

as $X_j^i = \emptyset$ so that, by (1), $B = B_i^i$, which is impossible, since for each i -open neighborhood $U(x)$: $U(x) \cap (B \setminus \{x\}) = U(x) \cap A \neq \emptyset$. \square

Note also here, that if $X_j^i \neq \emptyset$, then (1) (and so (2)) of Corollary 3.3 does not imply the (i, j) -submaximality of X .

Example 3.4. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. It is clear that $2\text{-}\mathcal{Bd}(X) = \{\{b\}, \{c\}, \{b, c\}\}$ and each of these sets is 1-discrete, while the set $A = \{a, b\} \in (1, 2)\text{-}\mathcal{LC}(X)$ so that (X, τ_1, τ_2) is not $(1, 2)$ -submaximal. The reason of this is that $X_2^1 = \{\{a\}\} \neq \emptyset$.

Corollary 3.5. *For a 1-T_1 BS $(X, \tau_1 < \tau_2)$ the following conditions are equivalent:*

- (1) (X, τ_1, τ_2) is $(2, 1)$ -submaximal.
- (2) Every subset of X is $\text{co}(2, 1)$ -locally closed.
- (3) If $A \in 1\text{-}\mathcal{Bd}(X)$, then $A \in \text{co } \tau_2$.
- (4) If $A \in 1\text{-}\mathcal{Bd}(X)$, then $A = A_2^i$.
- (5) $\tau_1 \text{ cl } A \setminus A \in \text{co } \tau_2$ for every subset $A \subset X$.
- (6) $\tau_1 \text{ cl } A \setminus A = (\tau_1 \text{ cl } A \setminus A)_2^i$ for every subset $A \subset X$.
- (7) If $A \in 1\text{-}\mathcal{D}(X)$, then $A \in \tau_2$.

Proof. Taking into account Theorem 3.2 and the first part of Corollary 3.3, it suffices to prove only that (6) \implies (5). Let $A \subset X$ be a set such that $\tau_1 \text{ cl } A \setminus A \notin \text{co } \tau_2$ so that there exists a point $p \in (\tau_1 \text{ cl } A \setminus A)_2^d \setminus (\tau_1 \text{ cl } A \setminus A)$. Then

$$p \in (\tau_1 \text{ cl } A \setminus A)_2^d \subset \tau_2 \text{ cl } (\tau_1 \text{ cl } A \setminus A) \subset \tau_2 \text{ cl } \tau_1 \text{ cl } A = \tau_1 \text{ cl } A.$$

Hence $p \in X \setminus \tau_1 \text{ cl } A$. On the other hand, $p \in (\tau_1 \text{ cl } A \setminus A)$ implies that $p \in (X \setminus \tau_1 \text{ cl } A) \cup A$ and, thus, $p \in A$. Let $B = A \setminus \{p\}$. Then $\tau_1 \text{ cl } A = \tau_1 \text{ cl } B \cup \{p\}$ as (X, τ_1, τ_2) is $1\text{-T}_1(\iff \text{R-}p\text{-T}_1)$. Moreover, $p \in (\tau_1 \text{ cl } A \setminus A)_2^d \subset (\tau_1 \text{ cl } A)_2^d \subset (\tau_1 \text{ cl } A)_1^d = A_1^d$ so that for any 1-open neighborhood $U(p) : U(p) \cap (A \setminus \{p\}) = U(p) \cap B \neq \emptyset$. Hence $p \in \tau_1 \text{ cl } B$ and, thus, $\tau_1 \text{ cl } B = \tau_1 \text{ cl } A$. Finally,

$$\tau_1 \text{ cl } B \setminus B = \tau_1 \text{ cl } A \setminus (A \setminus \{p\}) = (\tau_1 \text{ cl } A \setminus A) \cup \{p\}.$$

If there is a 2-open neighborhood $U(p)$ such that

$$U(p) \cap (\tau_1 \text{ cl } B \setminus B) = U(p) \cap ((\tau_1 \text{ cl } A \setminus A) \cup \{p\}) = \{p\},$$

then $U(p) \cap (\tau_1 \text{ cl } A \setminus A) = \emptyset$ which contradicts the assumption that $p \in \tau_2 \text{ cl } (\tau_1 \text{ cl } A \setminus A)$. \square

Corollary 3.6. Every BS (Y, τ'_1, τ'_2) of an (i, j) -submaximal BS (X, τ_1, τ_2) is also (i, j) -submaximal and if Y is j -discrete and $Y \cap X_j^i = \emptyset$, then Y is i -closed.

Proof. The first part is obvious. For the second part, by (3) of Theorem 3.2, it suffices to prove only that $\tau_j \text{ int } Y = \emptyset$. Contrary: $\tau_j \text{ int } Y \neq \emptyset$; then there are a point $x \in Y$ and a j -open neighborhood $U(x)$ such that $U(x) \subset Y$. Since $Y = Y_j^i$, there is a j -open neighborhood $V(x)$ such that $V(x) \cap Y = \{x\}$. Hence, $V(x) \cap U(x) = \{x\}$ which contradicts the condition $Y \cap X_j^i = \emptyset$. \square

Corollary 3.7. For a BS $(X, \tau_1 < \tau_2)$ we have:

- (1) If (X, τ_1, τ_2) is 1-submaximal, then it is d -nodec and p -nodec.
 - (2) If (X, τ_1, τ_2) is $(1, 2)$ -submaximal, then it is $(2, 1)$ -nodec.
- For a BS $(X, \tau_1 <_C \tau_2)$ we have
- (3) If (X, τ_1, τ_2) is $(1, 2)$ -submaximal, then it is 1-nodec.

For a BS $(X, \tau_1 <_N \tau_2)$ as well as for a BS $(X, \tau_1 <_S \tau_2)$ we have

(4) If (X, τ_1, τ_2) is $(1, 2)$ -submaximal, then it is d -nodec and p -nodec.

Proof. (1) Let $A \in i\text{-}\mathcal{ND}(X) \cup (i, j)\text{-}\mathcal{ND}(X)$. Then, in all cases, $A \in 1\text{-}\mathcal{Bd}(X)$ as $\tau_1 \subset \tau_2$. Hence, by (c) and (d) of Theorem 1.2 in [5], $A = \tau_1 \text{ cl } A = A_1^i$. But $\tau_1 \subset \tau_2$ implies that $A_1^i \subset A_2^i \subset A \subset \tau_2 \text{ cl } A \subset \tau_1 \text{ cl } A$ and so $A_1^i = A_2^i = A = \tau_2 \text{ cl } A = \tau_1 \text{ cl } A$.

Therefore, (X, τ_1, τ_2) is d -nodec and p -nodec.

(2) If $A \in (2, 1)\text{-}\mathcal{ND}(X)$, then $A \in 2\text{-}\mathcal{Bd}(X)$ and by (3) of Theorem 3.2, $A = \tau_1 \text{ cl } A$. Moreover, by the first part of Corollary 3.3, $A = A_1^i$ and, hence, $A = A_1^i = A_2^i = \tau_1 \text{ cl } A$ so that $A = \tau_1 \text{ cl } A = A_2^i$. Thus (X, τ_1, τ_2) is $(2, 1)$ -nodec.

(3) Let $A \in 1\text{-}\mathcal{ND}(X)$. Then, by (3) of Theorem 2.2.20 in [15], $A \in (2, 1)\text{-}\mathcal{ND}(X)$ and by the same reasonings as in the proof of (2), we obtain that $A = \tau_1 \text{ cl } A = A_1^i$, that is, (X, τ_1, τ_2) is 1-nodec.

(4) For the case where $\tau_1 <_S \tau_2$, we have (X, τ_1, τ_2) is $(1, 2)$ -submaximal $\iff (X, \tau_1, \tau_2)$ is 1-submaximal and, thus, it remains to use (1).

Let $\tau_1 <_N \tau_2$ and $A \in i\text{-}\mathcal{ND}(X) \cup (i, j)\text{-}\mathcal{ND}(X)$. Clearly, in all cases, $A \in (1, 2)\text{-}\mathcal{ND}(X)$ and according to (3) of Theorem 2.3.19 in [15], $A \in 2\text{-}\mathcal{ND}(X)$. Therefore, $A \in 2\text{-}\mathcal{Bd}(X)$ and since (X, τ_1, τ_2) is $(1, 2)$ -submaximal, by (3) of Theorem 3.2 and the first part of Corollary 3.3, $A = \tau_1 \text{ cl } A = A_1^i$. The rest is obvious. \square

Definition 3.8. A BS (X, τ_1, τ_2) is (i, j) -strongly σ -discrete if it can be represented as the union of a countable family of j -closed i -discrete subspaces, that is, if $X = \bigcup_{n=1}^{\infty} A_n$, where $A_n = \tau_j \text{ cl } A_n = (A_n)_i^i$ for each $n = \overline{1, \infty}$.

Clearly, for a BS $(X, \tau_1 < \tau_2)$ the following implications hold:

$$\begin{array}{ccc} (X, \tau_1, \tau_2) \text{ is } 1\text{-strongly } \sigma\text{-discrete} & \implies & (X, \tau_1, \tau_2) \text{ is } (2, 1)\text{-strongly } \sigma\text{-discrete} \\ \Downarrow & & \Downarrow \\ (X, \tau_1, \tau_2) \text{ is } (1, 2)\text{-strongly } \sigma\text{-discrete} & \implies & (X, \tau_1, \tau_2) \text{ is } 2\text{-strongly } \sigma\text{-discrete} . \end{array}$$

Now, it is not difficult to see that take place

Theorem 3.9. Every $(1, 2)$ -nodec BS $(X, \tau_1 < \tau_2)$ for which no nonempty 1-open subset is $(1, 2)$ -strongly σ -discrete, is a $(1, 2)$ -BrS, and hence, a 1-BrS.

Proof. Let $U \in \tau_1 \setminus \{\emptyset\}$ and U is of $(1, 2)$ -Catg I. Then, by Corollary 1.5.14 in [15], $U \in (1, 2)\text{-}\mathcal{Catg}_1(X)$ so that $U = \bigcup_{n=1}^{\infty} A_n$, where $A_n \in (1, 2)\text{-}\mathcal{ND}(X)$ for each $n = \overline{1, \infty}$. Evidently, each set A_n is 2-closed and 1-discrete, since (X, τ_1, τ_2) is $(1, 2)$ -nodec and, thus, U is $(1, 2)$ -strongly σ -discrete. The rest follows from (1) of Theorem 4.1.3 in [15]. \square

Corollary 3.10. For a BS $(X, \tau_1 <_C \tau_2)$ the following conditions are satisfied:

- (1) If (X, τ_1, τ_2) is $(1, 2)$ -nodec and a set $U \in \tau_1 \setminus \{\emptyset\}$ is $(1, 2)$ -strongly σ -discrete, then $\tau_2 \text{ cl } U (\iff \tau_1 \text{ cl } U)$ is 2-strongly σ -discrete.

- (2) If (X, τ_1, τ_2) is a $(2, 1)$ -nodec space for which no nonempty 1-open (2-open) subset is $(2, 1)$ -strongly σ -discrete, then it is a 1-BrS (an $A(2, 1)$ -BrS).

Proof. (1) Evidently, (X, τ_1, τ_2) is 2-nodec and if $U \in \tau_1 \setminus \{\emptyset\}$, U is $(1, 2)$ -strongly σ -discrete, then U is 2-strongly σ -discrete. Hence, $\tau_2 \text{ cl } U$ is also 2-strongly σ -discrete, since $U \in \tau_1 \subset \tau_2$. On the other hand, according to (3) of Corollary 2.2.7 in [15], $\tau_2 \text{ cl } U = \tau_1 \text{ cl } U$.

(2) Contrary: there is a set $U \in \tau_1 \setminus \{\emptyset\}$ such that U is of 1- Catg I ($\iff U \in 1\text{-Catg}_1(X)$). Then $U = \bigcup_{n=1}^{\infty} A_n$, where $A_n \in 1\text{-}\mathcal{ND}(X)$ and, hence, by (3) of Theorem 2.2.20 in [15], $A_n \in (2, 1)\text{-}\mathcal{ND}(X)$ for each $n = \overline{1, \infty}$. Therefore, (X, τ_1, τ_2) is $(2, 1)$ -nodec implies that $A_n = \tau_1 \text{ cl } A_n = (A_n)_2^i$ for each $n = \overline{1, \infty}$ and thus, U is $(2, 1)$ -strongly σ -discrete.

Now, taking into account (6) of Definition 1.1, the case of brackets is obvious. \square

Clearly, (2) of Corollary 3.10 and also Proposition 4.14 in [5] contain the sufficient conditions for a BS to be 1-Baire.

Very close to (1) of Corollary 3.10 is the following

Proposition 3.11. *If a p -normal BS $(X, \tau_1 <_C \tau_2)$ is $(1, 2)$ -nodec and a set $U \in \tau_1 \setminus \{\emptyset\}$ is $(1, 2)$ -strongly σ -discrete, then $\tau_2 \text{ cl } U$ ($\iff \tau_1 \text{ cl } U$) is also $(1, 2)$ -strongly σ -discrete.*

Proof. Let $U \in \tau_1 \setminus \{\emptyset\}$ and $U = \bigcup_{n=1}^{\infty} A_n$, where $A_n = \tau_2 \text{ cl } A_n = (A_n)_1^i$ for each $n = \overline{1, \infty}$. It is clear that $\tau_2 \text{ cl } U \setminus U \in (1, 2)\text{-}\mathcal{ND}(X)$.

Since (X, τ_1, τ_2) is $(1, 2)$ -nodec,

$$\tau_2 \text{ cl } U \setminus U = \tau_2 \text{ cl } (\tau_2 \text{ cl } U \setminus U) = (\tau_2 \text{ cl } U \setminus U)_1^i$$

so that

$$\tau_2 \text{ cl } U = U \cup (\tau_2 \text{ cl } U \setminus U) = \bigcup_{n=1}^{\infty} B_n,$$

where $B_n = A_n \cup (\tau_2 \text{ cl } U \setminus U)$ and

$$\tau_2 \text{ cl } B_n = \tau_2 \text{ cl } A_n \cup \tau_2 \text{ cl } (\tau_2 \text{ cl } U \setminus U) = A_n \cup (\tau_2 \text{ cl } U \setminus U) = B_n$$

for each $n = \overline{1, \infty}$. Hence, it remains to prove only that $B_n = (B_n)_1^i$, that is,

$$A_n \cup (\tau_2 \text{ cl } U \setminus U) = (A_n \cup (\tau_2 \text{ cl } U \setminus U))_1^i$$

for each $n = \overline{1, \infty}$. Since $A_n = (A_n)_1^i$ and $\tau_2 \text{ cl } U \setminus U = (\tau_2 \text{ cl } U \setminus U)_1^i$, it suffices to prove only that

$$(A_n \cup (\tau_2 \text{ cl } U \setminus U))_1^i = (B_n)_1^i = (A_n)_1^i \cup (\tau_2 \text{ cl } U \setminus U)_1^i$$

for each $n = \overline{1, \infty}$. The equality $A_n \cap (\tau_2 \text{ cl } U \setminus U) = \emptyset$ implies that $(B_n)_1^i \subset (A_n)_1^i \cup (\tau_2 \text{ cl } U \setminus U)_1^i$ for each $n = \overline{1, \infty}$. Indeed: for arbitrary $n \in N$ let $x \in (B_n)_1^i$; then there is a 1-open neighborhood $U(x)$ such that $U(x) \cap B_n = \{x\}$. Since

$B_n = A_n \cup (\tau_2 \text{ cl } U \setminus U)$, where $A_n \cap (\tau_2 \text{ cl } U \setminus U) = \emptyset$, we have $U(x) \cap A_n = \{x\}$ or $U(x) \cap (\tau_2 \text{ cl } U \setminus U) = \{x\}$ so that $x \in (A_n)_1^i$ or $x \in (\tau_2 \text{ cl } U \setminus U)_1^i$.

Finally, let us prove that $(A_n)_1^i \cup (\tau_2 \text{ cl } U \setminus U)_1^i \subset (B_n)_1^i$. First, let $x \in (A_n)_1^i$. Then there is a 1-open neighborhood $U(x)$ such that $U(x) \cap A_n = \{x\}$. Since $\tau_1 <_C \tau_2$ implies that $\tau_1 \text{ cl } U = \tau_2 \text{ cl } U$, we have that $V(x) = U(x) \setminus (\tau_2 \text{ cl } U \setminus U) \in \tau_1$, where $V(x) \cap B_n = V(x) \cap A_n = \{x\}$ so that $x \in (B_n)_1^i$. Now, let $x \in (\tau_2 \text{ cl } U \setminus U)_1^i$. Then there is a neighborhood $W(x) \in \tau_1$ such that $W(x) \cap (\tau_2 \text{ cl } U \setminus U) = \{x\}$.

Hence $x \in \overline{U} = \bigcup_{n=1}^{\infty} A_n$ so that $x \in \overline{A_n}$ for each $n = \overline{1, \infty}$. Since (X, τ_1, τ_2) is p -normal, $A_n \in \text{co } \tau_2$, $U \in \tau_1$ and $A_n \subset U$, there is a set $V_n \in \tau_1$ such that $A_n \subset V_n \subset \tau_2 \text{ cl } V_n = \tau_1 \text{ cl } V_n \subset U$, and $x \in \tau_1 \text{ cl } V_n$ for each $n = \overline{1, \infty}$. Putting now $V_n(x) = W(x) \setminus \tau_1 \text{ cl } V_n$ we obtain that $V_n(x) \in \tau_1$ and

$$\begin{aligned} V_n(x) \cap B_n &= (V_n(x) \cap A_n) \cup (V_n(x) \cap (\tau_2 \text{ cl } U \setminus U)) = \\ &= V_n(x) \cap (\tau_2 \text{ cl } U \setminus U) = \{x\} \end{aligned}$$

so that $x \in (B_n)_1^i$ for each $n = \overline{1, \infty}$. \square

Proposition 3.12. *For a $(2, 1)$ -submaximal BS $(X, \tau_1 < \tau_2)$ the following conditions are satisfied:*

- (1) *If (Y, τ'_1, τ'_2) is a BsS of X and $U = \tau_1 \text{ int } Y$, then the set $Z = Y \setminus Y_2^i \subset \tau_2 \text{ cl } U$ and $Y = A \cup B$, where $A \in \tau_1$ and $B = \tau_2 \text{ cl } B = B_2^i$.*
- (2) *If (C, τ'_1, τ'_2) is a 2-connected BsS of X , then $C \subset \tau_2 \text{ cl } \tau_1 \text{ int } C$.*

Proof. (1) Let

$$P = Y \setminus \tau_2 \text{ cl } U = Y \setminus \tau_2 \text{ cl } \tau_1 \text{ int } Y = Y \cap \tau_2 \text{ int } \tau_1 \text{ cl } (X \setminus Y).$$

Then $P \in 1\text{-}\mathcal{B}d(X)$ and by (3) and (4) of Corollary 3.5, $P = \tau_2 \text{ cl } P = P_2^i$. Since each point of P is 2-open in P and $P \in \tau'_2$, we have that $P \subset Y_2^i$ and, hence, $Z = Y \setminus Y_2^i \subset \tau_2 \text{ cl } U$. Clearly $Y = U \cup (Y \setminus U)$, where

$$\tau_1 \text{ int } (Y \setminus U) = \tau_1 \text{ int } (Y \cap (X \setminus \tau_1 \text{ int } Y)) = \emptyset$$

and, once more applying (3) and (4) of Corollary 3.5,

$$Y \setminus U = \tau_2 \text{ cl } (Y \setminus U) = (Y \setminus U)_2^i.$$

Thus $Y = A \cup B$, where $A = U \in \tau_1$ and $B = (Y \setminus U) = \tau_2 \text{ cl } B = B_2^i$ (clearly, if $\tau_1 \text{ int } Y = \emptyset$, then $Y = \tau_2 \text{ cl } Y = Y_2^i$ and $Y = \emptyset \cup Y$).

(2) First of all, let us note that $\tau_1 \text{ int } C \neq \emptyset$, since according to (4) of Corollary 3.5 the contrary means that $C = C_2^i$, that is, C is not 2-connected. If $C \setminus \tau_2 \text{ cl } \tau_1 \text{ int } C \neq \emptyset$, then

$$C = (C \cap \tau_2 \text{ cl } \tau_1 \text{ int } C) \cup (C \setminus \tau_2 \text{ cl } \tau_1 \text{ int } C),$$

where $A = C \cap \tau_2 \text{ cl } \tau_1 \text{ int } C \in \text{co } \tau'_2 \setminus \{\emptyset\}$ in (C, τ'_1, τ'_2) . It is clear that $C \setminus \tau_2 \text{ cl } \tau_1 \text{ int } C \in 1\text{-}\mathcal{B}d(X)$ and by (3) of Corollary 3.5, the set $B = C \setminus \tau_2 \text{ cl } \tau_1 \text{ int } C \in$

$\text{co } \tau'_2 \setminus \{\emptyset\}$. Hence $C = A \cup B$, where $A, B \in \text{co } \tau'_2 \setminus \{\emptyset\}$ and $A \cap B = \emptyset$. A contradiction with C is 2-connected. \square

Take place the following principal

Theorem 3.13. *Every 1-T₁, p -normal, p -connected and (2,1)-nodec BS $(X, \tau_1 <_C \tau_2)$ for which every 2-closed subset is (1,2)-WS-supernormal in X , is a (1,2)-BrS and, hence, a 1-BrS.*

To prove this theorem, we have to formulate

Lemma 3.14. *For a BS $(X, \tau_1 < \tau_2)$ the following conditions are satisfied:*

- (1) *If $F \in \text{co } \tau_1$, $\Phi \in \text{co } \tau_2$, $\Phi \subset F$ and the BsS (Φ, τ'_1, τ'_2) is p -WS-supernormal in X , then $(\Phi, \tau''_1, \tau''_2)$ is p -WS-supernormal in F .*
- (2) *If $F \in \text{co } \tau_1$, $F = \tau_1 \text{ cl } U$ for some $U \in \tau_1 \setminus \{\emptyset\}$ and $p\text{-ind } F = 0$, then (X, τ_1, τ_2) is p -disconnected.*

Proof. (1) First, let $A \in \text{co } \tau''_1$, $B \in \text{co } \tau'_2$ in (F, τ'_1, τ'_2) and $A \cap B = \emptyset$. Clearly $B \in \text{co } \tau_2$ and since $(\Phi, \tau''_1, \tau''_2)$ is (1,2)-WS-supernormal in X , there are sets $U \in \tau''_2$, $V' \in \tau_1$ such that $A \subset U$, $B \subset V'$ and $U \cap V' = \emptyset$. Since $B \subset F$, we have $B \subset F \cap V' = V$, where $V \in \tau'_1$. Therefore, $A \subset U$, $B \subset V$, where $U \in \tau''_2$, $V \in \tau'_1$ and $U \cap V = \emptyset$ so that $(\Phi, \tau''_1, \tau''_2)$ is (1,2)-WS-supernormal in F .

Now, let $A \in \text{co } \tau''_2$, $B \in \text{co } \tau'_1$ and $A \cap B = \emptyset$. Because $B \in \text{co } \tau_1$ and $(\Phi, \tau''_1, \tau''_2)$ is (2,1)-WS-supernormal in X , there are sets $U \in \tau''_1$, $V' \in \tau_2$ such that $A \subset U$, $B \subset V'$ and $U \cap V' = \emptyset$. But $B \subset F$ implies that $B \subset F \cap V' = V \in \tau'_2$ so that $A \subset U$, $B \subset V$, where $U \in \tau''_1$, $V \in \tau'_2$ and $U \cap V = \emptyset$. Thus $(\Phi, \tau''_1, \tau''_2)$ is (2,1)-WS-supernormal in F .

(2) $p\text{-ind } F = 0 \iff ((1,2)\text{-ind } F = 0 \text{ and } (2,1)\text{-ind } F = 0)$. Evidently, $(1,2)\text{-ind}_x F = 0$ for each point $x \in U$, where $\tau_1 \text{ cl } U = F$ and $U \in \tau_1 \setminus \{\emptyset\}$. Let $U''(x) \in \tau'_1$ be any neighborhood of an arbitrarily fixed point $x \in U$ in the BsS (F, τ'_1, τ'_2) . Then there is a neighborhood $U'(x) \in \tau_1$ such that $U'(x) \cap F = U''(x)$. If $U(x) = U'(x) \cap U$, then $U(x) \in \tau_1$ and since $U(x) \subset F$, we have $U(x) \in \tau'_1$. Since $(1,2)\text{-ind}_x F = 0$, by (1) of Corollary 3.1.6 in [15], there is a neighborhood $V(x) \in \tau'_1$ such that $V(x) \subset U(x)$ and $\tau'_2 \text{ cl } V(x) \setminus V(x) = \emptyset$. But for $V(x) \in \tau'_1$ there is $V'(x) \in \tau_1$ such that $V'(x) \cap F = V(x)$. Because $U \subset F$ and $V(x) \subset U(x) \subset U$, we have $V(x) = V'(x) \cap U$ and so $V(x) \in \tau_1$. Moreover, $F \in \text{co } \tau_1 \subset \text{co } \tau_2$ and so $V(x) = \tau'_2 \text{ cl } V(x) = \tau_2 \text{ cl } V(x)$. Therefore, $V(x) \in \tau_1 \cap \text{co } \tau_2$, $\emptyset \neq V(x) \neq X$ and by (c) of Theorem A in [18], (X, τ_1, τ_2) is p -disconnected. \square

Now, we can proceed to prove Theorem 3.13.

Following Theorem 3.9 and Proposition 3.11, it suffices to prove only that if $U \in \tau_1 \setminus \{\emptyset\}$, then $\tau_1 \text{ cl } U$ is not (1,2)-strongly σ -discrete. Contrary: there is a set $U \in \tau_1 \setminus \{\emptyset\}$ such that $\tau_1 \text{ cl } U = F$ is (1,2)-strongly σ -discrete. Then $F = \bigcup_{k=1}^{\infty} F_k$, where $F_k = \tau_2 \text{ cl } F_k = (F_k)_1^i$ and since $\tau_1 \subset \tau_2$, we have $F_k = (F_k)_2^i$ so that $p\text{-Ind } F_k = 0$ for each $k = \overline{1, \infty}$. Furthermore, $F_k \in \text{co } \tau_2 \subset p\text{-Cl}(X)$ and $F_k \subset F$ imply that $F_k \in \text{co } \tau'_2 \subset p\text{-Cl}(F)$ in (F, τ'_1, τ'_2) for each $k = \overline{1, \infty}$. Since (X, τ_1, τ_2) is p -normal, by Corollary 3.2.6 in [15], (F, τ'_1, τ'_2) is also p -normal as $F \in \text{co } \tau_1 \subset p\text{-Cl}(X)$. Moreover, since each F_k is 2-closed in X , by the remark

between Definition 2.16 and Example 2.17, each F_k is $(2, 1)$ -WS-supernormal in X and so, by the assumption of this theorem and (1) of Lemma 3.14, each F_k is p -WS-supernormal in Y . Therefore, Corollary 2.25 gives that $p\text{-Ind } F = 0$ and, hence, by the second part of Corollary 3.2.8 in [15], $p\text{-ind } F = 0$ as X is 1-T_1 . Thus, by (2) of Lemma 3.14, (X, τ_1, τ_2) is p -disconnected, which contradicts the assumption. \square

Corollary 3.15. *Every 1-T_1 , p -normal, p -connected and 1 -submaximal (or $(1, 2)$ -submaximal) BS $(X, \tau_1 <_C \tau_2)$ for which every 2 -closed subset is $(1, 2)$ -WS-supernormal in X , is a $(1, 2)$ -BrS and, hence, a 1 -BrS.*

Proof. Follows directly from (1) (or (2)) of Corollary 3.7 and Theorem 3.13. \square

Definition 3.16. A BS (X, τ_1, τ_2) is an (i, j) - I -space if its j -derived set is j -closed and i -discrete, that is, if $X_j^d = \tau_j \text{ cl } X_j^d = (X_j^d)_i^i$.

Proposition 3.17. *For a 1-T_1 , BS $(X, \tau_1 < \tau_2)$ the following implications hold:*

$$\begin{array}{ccc} (X, \tau_1, \tau_2) \text{ is a } 1\text{-}I\text{-space} & \implies & (X, \tau_1, \tau_2) \text{ is a } (2, 1)\text{-}I\text{-space} \\ \Downarrow & & \Downarrow \\ (X, \tau_1, \tau_2) \text{ is a } (1, 2)\text{-}I\text{-space} & \implies & (X, \tau_1, \tau_2) \text{ is a } 2\text{-}I\text{-space.} \end{array}$$

Proof. Evidently, $X_1^d = \tau_1 \text{ cl } X_1^d = (X_1^d)_1^i$ implies that $X_1^d = \tau_1 \text{ cl } X_1^d = (X_1^d)_2^i$ and $X_2^d = \tau_2 \text{ cl } X_2^d = (X_2^d)_1^i$ implies that $X_2^d = \tau_2 \text{ cl } X_2^d = (X_2^d)_2^i$ so that the horizontal implications hold. Moreover, $X_1^d = (X_1^d)_1^i$ implies that $X_2^d = (X_2^d)_1^i$ and $X_1^d = (X_1^d)_2^i$ implies that $X_2^d = (X_2^d)_2^i$. But (X, τ_1, τ_2) is also 2-T_1 so that $X_2^d = \tau_2 \text{ cl } X_2^d$ and we obtain the vertical implications. \square

Remark 3.18. If a nonempty BS (X, τ_1, τ_2) is an i - I -space, then $X \in i\text{-DI}(X)$, since by [5, p. 221], $X \in i\text{-ST}(X)$.

Hence, if a nonempty BS $(X, \tau_1 < \tau_2)$ is an i - I -space or an (i, j) - I -space, then $X \in 2\text{-DI}(X)$. But, if a BS $(X, \tau_1 < \tau_2)$ is a $(2, 1)$ - I -space and $X \in 1\text{-DI}(X) \setminus 2\text{-DI}(X)$ ($\iff X \in p\text{-DI}(X) \setminus 2\text{-DI}(X)$ by (2) of Proposition 1.4.2 in [6]), then $X = X_2^i$. Indeed: $X \in 1\text{-DI}(X) \setminus 2\text{-DI}(X)$ implies that $X_2^d \neq X = X_1^d$ and since (X, τ_1, τ_2) is a $(2, 1)$ - I -space, we have: $X = X_1^d = (X_1^d)_2^i = X_2^i$.

According to Proposition 3.17 above and Proposition 1.5 in [5] it is also evident that every 1-T_1 BS $(X, \tau_1 < \tau_2)$ with only finitely many 1 -nonisolated points is an i - I -space and an (i, j) - I -space.

Proposition 3.19 as well as (2) of Proposition 3.20 and Corollaries 3.21, 3.22 describes the connections between topological and bitopological versions of submaximal, nodec and I -spaces.

Proposition 3.19. *In the class of 1 -scattered ($\iff p$ -scattered) BS's of the type $(X, \tau_1 < \tau_2)$ the following conditions are equivalent:*

- (1) (X, τ_1, τ_2) is a 1 - I -space.
- (2) (X, τ_1, τ_2) is d -submaximal and p -submaximal.
- (3) (X, τ_1, τ_2) is 1 -nodec.

Proof. In the class of 1-scattered BS's of the type $(X, \tau_1 < \tau_2)$ the equivalences: (X, τ_1, τ_2) is a 1- I -space $\iff (X, \tau_1, \tau_2)$ is 1-submaximal $\iff (X, \tau_1, \tau_2)$ is 1-nodect, are given by Corollary 1.8 in [5]. The rest follows directly from the implications after Definition 3.1. \square

Proposition 3.20. *The following conditions are satisfied:*

- (1) *A j -T₁ BS (X, τ_1, τ_2) is an (i, j) - I -space if and only if $A_j^d = (A_j^d)_i^i$ for each subset $A \subseteq X$.*
- (2) *If (X, τ_1, τ_2) is (i, j) -nodect and $X_j^i \in i\text{-}\mathcal{D}(X)$, then (X, τ_1, τ_2) is an (i, j) - I -space.*

Proof. (1) The implication from right to left is obvious. Let X be an (i, j) - I -space and let $A \subset X$ be any subset. Then it remains to prove only that $A_j^d \subset (A_j^d)_i^i$. If $x \in A_j^d$, then $x \in X_j^d = (X_j^d)_i^i$ and, hence, there is a neighborhood $U(x) \in \tau_i$ such that $U(x) \cap X_j^d = \{x\}$. Since $x \in A_j^d$, it is clear that $U(x) \cap A_j^d = \{x\}$ and so $x \in (A_j^d)_i^i$.

(2) Clearly, $\tau_i \text{ cl } X_j^i = X$ implies that $\tau_i \text{ int } X_j^d = \emptyset$ and so $X_j^d \in (i, j)\text{-}\mathcal{ND}(X)$, since $X_j^i \in \tau_j$. Thus $X_j^d = \tau_j \text{ cl } X_j^d = (X_j^d)_i^i$ as (X, τ_1, τ_2) is (i, j) -nodect. \square

Corollary 3.21. *Every 1-T₁ and $(2, 1)$ - I -space $(X, \tau_1 < \tau_2)$ is $(2, 1)$ -submaximal and, hence, 2-nodect.*

Proof. Let $A \subset X$ be any set. Then, by (1) of Proposition 3.20,

$$\tau_1 \text{ cl } A \setminus A \subset A_1^d = (A_1^d)_2^i$$

and so $\tau_1 \text{ cl } A \setminus A = (\tau_1 \text{ cl } A \setminus A)_2^i$. Hence, by (6) of Corollary 3.5, (X, τ_1, τ_2) is $(2, 1)$ -submaximal and by the implications after Definition 3.1, (X, τ_1, τ_2) is 2-submaximal. Thus, it remains to use Corollary 1.3 in [5]. \square

Corollary 3.22. *Every 1-T₁ and $(1, 2)$ -submaximal BS $(X, \tau_1 < \tau_2)$, for which $X_1^i \in 2\text{-}\mathcal{D}(X)$, is a $(2, 1)$ - I -space, a 2- I -space and a 2-nodect space.*

Proof. Indeed, by (2) of Corollary 3.7, (X, τ_1, τ_2) is $(2, 1)$ -nodect. Hence, it remains to use (2) of Proposition 3.20, Proposition 3.17 and Corollary 1.3 in [5], since (X, τ_1, τ_2) is 2-submaximal. \square

Taking into account (2) of Proposition 3.20 and Theorem 1.6 in [5], Proposition 3.17 implies that if a 1-T₁ BS $(X, \tau_1 < \tau_2)$ is an i - I -space or an (i, j) - I -space, then $X_2^i \in 2\text{-}\mathcal{D}(X) \subset 1\text{-}\mathcal{D}(X)$. Moreover, as we mentioned above (see Remark 3.18), under the same hypotheses, $X \overline{\in} 2\text{-}\mathcal{DI}(X)$. By Theorem 1.6 in [5], for a 1-T₁ and 1- I -space $(X, \tau_1 < \tau_2)$, the set $X_1^i \in 1\text{-}\mathcal{D}(X)$. In this context take place

Proposition 3.23. *If for a 1-T₁ and $(2, 1)$ - I -space $(X, \tau_1 < \tau_2)$ we have $X_1^i \in 1\text{-}\mathcal{D}(X) \setminus 2\text{-}\mathcal{D}(X)$, then $\tau_2 \text{ int } X_1^d \subset X_2^i$ and $X \overline{\in} 1\text{-}\mathcal{DI}(X)$ ($\iff X \overline{\in} p\text{-}\mathcal{DI}(X)$).*

Proof. First of all, note that $X \overline{\in} 1\text{-}\mathcal{DI}(X)$ is evident, and by (2) of Proposition 1.4.2 in [15], $X \overline{\in} p\text{-}\mathcal{DI}(X)$. Furthermore, $X_1^i \in 2\text{-}\mathcal{D}(X)$ implies that $X_1^d \overline{\in} 2\text{-}\mathcal{BD}(X)$ and since (X, τ_1, τ_2) is a $(2, 1)$ - I -space, we have $\emptyset \neq \tau_2 \text{ int } X_1^d =$

$\tau_2 \text{int}(X_1^d)_2^i$. Hence, if $x \in \tau_2 \text{int} X_1^d$ is any point, then there are neighborhoods $U(x)$, $V(x) \in \tau_2$ such that $U(x) \subset X_1^d$ and $V(x) \cap X_1^d = \{x\}$. Evidently,

$$U(x) \cap V(x) = (U(x) \cap X_1^d) \cap V(x) = U(x) \cap (X_1^d \cap V(x)) = \{x\} \in \tau_2$$

and so $x \in X_2^i$. \square

Proposition 3.24. *Every 1-T₁ BS $(X, \tau_1 < \tau_2)$ with only finitely many 1-nonisolated points is a d -I-space, a p -I-space, d -submaximal, p -submaximal, d -nodec, p -nodec, d -scattered and p -scattered.*

Proof. Indeed, by Proposition 1.5 in [5], (X, τ_1, τ_2) is a 1-I-space. Hence, by Proposition 3.17, (X, τ_1, τ_2) is a d -I-space and a p -I-space. Further, by Corollary 1.3 in [5], (X, τ_1, τ_2) is 1-submaximal and, hence, by the implications after Definition 3.1, it is d -submaximal and p -submaximal. Since (X, τ_1, τ_2) is 1-submaximal, by (1) of Corollary 3.7, it is d -nodec and p -nodec. Finally, by [5, p. 221], (X, τ_1, τ_2) is d -scattered, since (X, τ_1, τ_2) is a d -I-space and, therefore, by (3) of Proposition 1.4.12 in [15], (X, τ_1, τ_2) is p -scattered. \square

Proposition 3.25. *For a BS $(X, \tau_1 < \tau_2)$ the following conditions are satisfied:*

- (1) *If $(X, \tau_1 <_C \tau_2)$ is $(1, 2)$ -submaximal, then $X = Y \cup Z$, where $Y \in \text{co } \tau_2$, (Y, τ'_1, τ'_2) is a $(2, 1)$ -I-space and $Z \in \text{co } \tau_1 \cap 1\text{-DI}(Z)$, (Z, τ''_1, τ''_2) is $(1, 2)$ -submaximal.*
- (2) *If (X, τ_1, τ_2) is $(1, 2)$ -nodec, then $X = Y \cup Z$, where $Y \in \text{co } \tau_1$, (Y, τ'_1, τ'_2) is a $(1, 2)$ -I-space and $Z \in \text{co } \tau_2 \cap 2\text{-DI}(Z)$, (Z, τ''_1, τ''_2) is $(1, 2)$ -nodec.*

Proof. (1) It is clear that $X_1^i \in 2\text{-D}(\tau_2 \text{cl } X_1^i)$. If $Y = \tau_2 \text{cl } X_1^i$, then by Corollary 3.6, (Y, τ'_1, τ'_2) is also $(1, 2)$ -submaximal and, hence, by (2) of Corollary 3.7, (Y, τ'_1, τ'_2) is $(2, 1)$ -nodec. On the other hand, $Y_1^i \in 2\text{-D}(Y)$ and according to (2) of Proposition 3.20, (Y, τ'_1, τ'_2) is a $(2, 1)$ -I-space. Furthermore, since $\tau_1 <_C \tau_2$ and $X_1^i \in \tau_1$, by (3) of Corollary 2.2.7 in [15], $Y = \tau_2 \text{cl } X_1^i = \tau_1 \text{cl } X_1^i$. Therefore, $X \setminus Y = X \setminus \tau_1 \text{cl } X_1^i \in \tau_1$ and, hence, $(X \setminus Y) \cap X_1^i = \emptyset$ implies that $(X \setminus Y)_1^i = \emptyset$, so that $(\tau_1 \text{cl}(X \setminus Y))_1^i = \emptyset$.

Clearly $X = Y \cup Z$, where $Z = \tau_1 \text{cl}(X \setminus Y)$ and once more applying Corollary 3.6, we obtain that (Z, τ''_1, τ''_2) is also $(1, 2)$ -submaximal, $Z \in \text{co } \tau_1 \cap 1\text{-DI}(Z)$.

(2) Evidently, $X_2^i \in 1\text{-D}(\tau_1 \text{cl } X_2^i)$. If $Y = \tau_1 \text{cl } X_2^i$, then by Remark 1.2, (Y, τ'_1, τ'_2) is also $(1, 2)$ -nodec. Hence, according to (2) of Proposition 3.20, (Y, τ'_1, τ'_2) is $(1, 2)$ -I-space, since $Y_2^i \in 1\text{-D}(Y)$. Clearly $X \setminus Y \in \tau_1 \subset \tau_2$ and since $(X \setminus Y) \cap X_2^i = \emptyset$, we have $(X \setminus Y)_2^i = \emptyset$.

Therefore, $(\tau_2 \text{cl}(X \setminus Y))_2^i = \emptyset$ and if $Z = \tau_2 \text{cl}(X \setminus Y)$, then $Z \in \text{co } \tau_2 \cap 2\text{-DI}(Z)$ and, by Remark 1.2, (Z, τ''_1, τ''_2) is $(1, 2)$ -nodec. \square

Definition 3.26. A surjection $f : (X, \tau_1, \tau_2) \rightarrow (Y, \gamma_1, \gamma_2)$ is (i, j) -locally closed quotient (briefly, (i, j) -lcq), if $f^{-1}(A) \in (i, j)\text{-LC}(X)$ implies that $A \in (i, j)\text{-LC}(Y)$ or, equivalently, if $f^{-1}(A) \in \text{co}(i, j)\text{-LC}(X)$ implies that $A \in \text{co}(i, j)\text{-LC}(Y)$ for each subset $A \subset Y$.

Proposition 3.27. *For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \gamma_1, \gamma_2)$ the following conditions are satisfied:*

- (1) If f is a d -open or a d -closed surjection, then f is (i, j) -lcq.
- (2) If f is (i, j) -lcq and (X, τ_1, τ_2) is (i, j) -submaximal, then (Y, γ_1, γ_2) is also (i, j) -submaximal.

Proof. (1) It suffices to consider only the case where f is a d -open surjection. Let $B \subset Y$ and $f^{-1}(B) \in \text{co}(i, j)\text{-}\mathcal{LC}(X)$, that is, $f^{-1}(B) = A_1 \cup A_2$, where $A_1 \in \tau_j$ and $A_2 \in \text{co } \tau_i$. Then $B_1 = f(A_1) \in \gamma_j$, $B_2 = Y \setminus f(X \setminus A_2) \in \text{co } \gamma_i$ and $B = B_1 \cup B_2$.

(2) Let $A \subset Y$ be any subset. Since (X, τ_1, τ_2) is (i, j) -submaximal, $f^{-1}(A) \in (i, j)\text{-}\mathcal{LC}(X)$ and by Definition 3.26, $A \in (i, j)\text{-}\mathcal{LC}(Y)$ so that (Y, γ_1, γ_2) is (i, j) -submaximal. \square

Corollary 3.28. If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \gamma_1, \gamma_2)$ is a d -open or a d -closed surjection and (X, τ_1, τ_2) is p -submaximal, then (Y, γ_1, γ_2) is also p -submaximal.

Corollary 3.29. If (X, τ_1, τ_2) is p -submaximal and $\tau_i \subseteq \gamma_i$, then (X, γ_1, γ_2) is also p -submaximal.

Proof. Indeed, the identity map $i_x : (X, \tau_1, \tau_2) \rightarrow (X, \gamma_1, \gamma_2)$ is a d -open and d -closed surjection. \square

Theorem 3.30. If a BS $(X, \tau_1 < \tau_2)$ is a $(2, 1)$ - I -space and $f : (X, \tau_1 < \tau_2) \rightarrow (Y, \gamma_1 < \gamma_2)$ is a d -closed or a d -open surjection, then $(Y, \gamma_1 < \gamma_2)$ is also a $(2, 1)$ - I -space.

Proof. First, let f be a d -closed surjection. Since (X, τ_1, τ_2) is a $(2, 1)$ - I -space, for the set $F = X_1^d = \tau_1 \text{ cl } X_1^d = (X_1^d)_2^i$ the BS $(F, \tau'_1 < \tau'_2)$ is 2-discrete. Hence, $2^F \subset \text{co } \tau'_2 \subset \text{co } \tau_2$ and, also, $\text{co } \tau'_1 \subset \text{co } \tau_1$ as $F \in \text{co } \tau_1 \subset \text{co } \tau_2$. Therefore, the restriction $f|_F : (F, \tau'_1, \tau'_2) \rightarrow (Y, \gamma_1, \gamma_2)$ is d -closed. Let $P = f(F) \subset Y$. Evidently, for each subset $B \subseteq P$ there is a subset $A \subseteq F$ such that $f(A) = B$ so that each subset $B \subseteq P$ is 2-closed in (Y, γ_1, γ_2) as each subset $A \subseteq F$ is 2-closed in (X, τ_1, τ_2) . Hence, the BS $(P, \gamma'_1 < \gamma'_2)$ is also 2-discrete so that $P = \gamma_1 \text{ cl } P = P_2^i$ as $P = f(F)$, $F \in \text{co } \tau_1$ and f is 1-closed. Therefore, to completes the proof of the first part, it suffices to prove only that each point $x \in Y \setminus P$ is 1-isolated in (Y, γ_1, γ_2) .

Let $x \in Y \setminus P$ be any point. Then $E = f^{-1}(x)$ implies that $E \subset X \setminus F = X \setminus X_1^d = X_1^i$. Hence, $E \in \tau_1$ and so $\Phi = X \setminus E \in \text{co } \tau_1$. Because f is 1-closed, the set $f(\Phi) \in \text{co } \gamma_1$ and $\{x\} = Y \setminus f(\Phi)$, that is, $\{x\} \in \gamma_1$ so that $x \in Y_1^i$. Therefore, $Y_1^d = P$ and, hence, $Y_1^d = \tau_1 \text{ cl } Y_1^d = (Y_1^d)_2^i$, that is, (Y, γ_1, γ_2) is a $(2, 1)$ - I -space.

For a d -open f , if $U = X_1^i$, then $U \in \tau_1$ and, therefore, $f(U) \in \gamma_1$, $f(U) \subset Y_1^i$. Putting now $P = Y \setminus f(U)$ we obtain that $P \in \text{co } \gamma_1$ and $f^{-1}(P) \subset F$, where $F = X \setminus U \in \text{co } \tau_1$. Since (X, τ_1, τ_2) is a $(2, 1)$ - I -space, for the set $F = X_1^d$, we have $F = \tau_1 \text{ cl } F = F_2^i$. Therefore, all subsets of F are 2-closed in F and, hence, in X . Since $f^{-1}(P) \subset F$, for each subset $B \subseteq P$ we have $f^{-1}(B) \in \text{co } \tau_2$. Hence, for each subset $B \subseteq P$ the set $X \setminus f^{-1}(B) \in \tau_2$ and since f is 2-open, the set $f(X \setminus f^{-1}(B)) = Y \setminus B \in \gamma_2$. Therefore, $P = \gamma_1 \text{ cl } P = P_2^i$, where each point of $Y \setminus P = f(U)$ is 1-isolated in Y . Thus $Y_1^d = P$ and $Y_1^d = \tau_1 \text{ cl } Y_1^d = (Y_1^d)_2^i$ so that (Y, γ_1, γ_2) is a $(2, 1)$ - I -space. \square

The rest of this section is concerned with the notion of D -space and its bitopological modifications. The sufficient conditions are established under which a TS is a D -space on the one hand, by the topological methods and, on the other hand, by the bitopological ones.

Definition 3.31. A neighborhood assignment on a TS (X, τ) is a function $\phi : X \rightarrow \tau$ such that $x \in \phi(x)$. A TS (X, τ) is a D -space if for every neighborhood assignment ϕ on X there is a closed discrete subset $D \subseteq X$ such that $\bigcup_{x \in D} \phi(x) = X$ [21].

Remark 3.32. In general, a D -space is not compact. Indeed, if $\phi : X \rightarrow \tau$ is any neighborhood assignment on X , then there is a subset $D = \overline{D} = D^i$ such that $\bigcup_{x \in D} \phi(x) = X$. Let us consider the open covering \mathcal{U} of the TS (X, τ) :

$$\mathcal{U} = \{\phi(x) \setminus D : x \in X \setminus D\} \cup \{U(x) : U(x) \in \tau, U(x) \cap D = \{x\}\}.$$

Clearly, if D is infinite, then \mathcal{U} does not contain a finite subcovering.

Let $\phi : X \rightarrow \tau$ be a neighborhood assignment on the TS (X, τ) and let $f : (X, \tau) \rightarrow (Y, \gamma)$ be an open surjection. Then $\psi : Y \rightarrow \gamma$, defined as $\psi(y) = f(\bigcup_{x \in f^{-1}(y)} \phi(x))$, is the neighborhood assignment on the TS (Y, γ) . Also, if $\psi : Y \rightarrow \gamma$ is a neighborhood assignment on the TS (Y, γ) and $f : (X, \tau) \rightarrow (Y, \gamma)$ is any continuous function, then $\phi : X \rightarrow \tau$, defined as $\phi(x) = f^{-1}(\psi(f(x)))$, is the neighborhood assignment on the TS (X, τ) .

Definition 3.33. Let $\phi : X \rightarrow \tau$ and $\psi : Y \rightarrow \gamma$ be neighborhood assignments on the TS's (X, τ) and (Y, γ) , respectively. Then we shall say that a surjection $f : X \rightarrow Y$ connects ϕ with ψ if $\psi \circ f = f \circ \phi$, i.e., if $\psi(f(x)) = f(\phi(x))$ for each point $x \in X$.

Remark 3.34. (a) If $\phi : X \rightarrow \tau$ and $\psi : Y \rightarrow \gamma$ are neighborhood assignments on the TS's (X, τ) and (Y, γ) , respectively, where ψ is defined by an open bijection $f : (X, \tau) \rightarrow (Y, \gamma)$, then f connects ϕ with ψ .

Indeed,

$$\psi(f(x)) = f\left(\bigcup_{x \in f^{-1}(f(x))} \phi(x)\right) = f(\phi(x)) \text{ for each } x \in X.$$

(b) If $\phi : X \rightarrow \tau$ and $\psi : Y \rightarrow \gamma$ are neighborhood assignments on the TS's (X, τ) and (Y, γ) , respectively, where ψ is defined by a continuous surjection $f : (X, \tau) \rightarrow (Y, \gamma)$, then f connects ϕ with ψ .

Indeed,

$$f(\phi(x)) = f(f^{-1}(\psi(f(x)))) = \psi(f(x)) \text{ for each } x \in X.$$

It is clear that if a surjection $f : X \rightarrow Y$ connects a neighborhood assignment $\phi : X \rightarrow \tau$ with a neighborhood assignment $\psi : Y \rightarrow \gamma$, then $f(\phi(x)) \in \gamma$ for each point $x \in X$; but, in general, f is not open.

Example 3.35. Let $X = \{a, b, c, d, e\}$, τ be the discrete topology on X and let a neighborhood assignment $\phi : X \rightarrow \tau$ on the TS (X, τ) be defined in the manner as follows:

$$\phi(a) = \{a\}, \quad \phi(b) = \{a, b\}, \quad \phi(c) = \{a, b, c\}, \quad \phi(d) = \{a, b, c, d\}$$

and $\phi(e) = X$. Furthermore, let $Y = \{0, 1, 2, 3, 4\}$,

$$\gamma = \{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\}, \{0, 1, 2, 3\}, Y\}$$

and let a neighborhood assignment $\psi : Y \rightarrow \gamma$ on the TS (Y, γ) be defined as follows:

$$\begin{aligned} \psi(0) &= \{0\}, \quad \psi(1) = \{0, 1\}, \quad \psi(2) = \{0, 1, 2\}, \\ \psi(3) &= \{0, 1, 2, 3\} \quad \text{and} \quad \psi(4) = Y. \end{aligned}$$

If $f : X \rightarrow Y$ is defined as follows:

$$f(a) = 0, \quad f(b) = 1, \quad f(c) = 2, \quad f(d) = 3 \quad \text{and} \quad f(e) = 4,$$

then f is a continuous bijection (i.e., a compression map), as τ is discrete; moreover, f connects ϕ with ψ . But, f is not open, since, for example, $f(a, c) = \{0, 2\} \not\subseteq \gamma$, where $\{a, c\} \in \tau$, and so f is not a homeomorphism.

Now, if replace the sets X and Y and if we consider the map $f^{-1} : Y \rightarrow X$, then f^{-1} will become an open bijection, connecting ψ with ϕ . Clearly, f^{-1} is not homeomorphism, since it is not continuous.

Moreover, it follows immediately from the determination of the topology γ on the set Y that for any neighborhood assignment $\psi : Y \rightarrow \gamma$ always $\psi(4) = Y$, since the set $\{0, 1, 2, 3, 4\} = Y$ is a unique set from γ which contains the point 4. Hence, for any neighborhood assignment $\psi : Y \rightarrow \gamma$ we can associate the closed discrete set $\{4\}$ such that $\psi(4) = Y$. Thus (Y, γ) is a D -space.

Note also here that if a compression map $g : X \rightarrow Y$ is defined as follows: $g(a) = 1, g(b) = 0, g(c) = 2, g(d) = 3$ and $g(e) = 4$, then g does not connect ϕ with ψ since $\psi(g(a)) = \psi(1) = \{0, 1\} \neq g(\phi(a)) = g(a) = \{1\}$.

Theorem 3.36. *The following conditions are satisfied:*

- (1) *If for each neighborhood assignment $\phi : X \rightarrow \tau$ on the TS (X, τ) there are a neighborhood assignment $\psi : Y \rightarrow \gamma$ on the TS (Y, γ) and a compression map $f : (X, \tau) \rightarrow (Y, \gamma)$ which connects ϕ with ψ , then (Y, γ) is a D -space implies that (X, τ) is also a D -space.*
- (2) *If for each neighborhood assignment $\psi : Y \rightarrow \gamma$ on the TS (Y, γ) there are a neighborhood assignment $\phi : X \rightarrow \tau$ on the TS (X, τ) and an open bijection $f : (X, \tau) \rightarrow (Y, \gamma)$ which connects ϕ with ψ , then (X, τ) is a D -space implies that (Y, γ) is also a D -space.*

Proof. (1) Let $\phi : X \rightarrow \tau$ be any neighborhood assignment on the TS (X, τ) . Then there are a neighborhood assignment $\psi : Y \rightarrow \gamma$ on the TS (Y, γ) and a

compression map $f : (X, \tau) \rightarrow (Y, \gamma)$ such that $\psi \circ f = f \circ \phi$. Since (Y, γ) is a D -space, for ψ there exists a closed discrete set D' such that $\bigcup_{y \in D'} \psi(y) = Y$.

Clearly $f^{-1}(D') = D = \overline{D}$ as f is continuous. On the other hand, if $x \in D$ is any point, then $y = f(x) \in D'$ and so there is a neighborhood $U(y) \in \gamma$ such that $U(y) \cap D' = \{y\}$. Since f is a compression map,

$$x = f^{-1}(y) = f^{-1}(U(y) \cap D') = f^{-1}(U(y)) \cap f^{-1}(D') = U(x) \cap D,$$

where $U(x) \in \tau$. Hence, D is closed discrete subset of (X, τ) . Finally, $\bigcup_{y \in D'} \psi(y) = Y$ implies that

$$\begin{aligned} X &= f^{-1}(Y) = \\ &= f^{-1}\left(\bigcup_{y \in D'} \psi(y)\right) = \bigcup_{x \in D} f^{-1}(\psi(f(x))) = \bigcup_{x \in D} f^{-1}(f(\phi(x))) = \bigcup_{x \in D} \phi(x) \end{aligned}$$

as f connects ϕ with ψ .

(2) If $\psi : Y \rightarrow \gamma$ is any neighborhood assignment on the TS (Y, τ) , then there are a neighborhood assignment $\phi : X \rightarrow \tau$ on the TS (X, τ) and an open bijection $f : (X, \tau) \rightarrow (Y, \gamma)$ which connects ϕ with ψ . Since (X, τ) is a D -space, there is a closed discrete set D such that $\bigcup_{x \in D} \phi(x) = X$. If $U = X \setminus D$, then $D' = f(D) = f(X \setminus U) = Y \setminus f(U) \in \text{co } \gamma$. For any point $y \in D'$ there is a neighborhood $U(x) \in \tau$ of the point $x = f^{-1}(y)$ such that $U(x) \cap D = \{x\}$. Hence

$$\begin{aligned} y &= f(f^{-1}(y)) = f(x) = f(U(x) \cap D) = \\ &= f(U(x) \cap f^{-1}(D')) = f(U(x)) \cap D' = U(y) \cap D', \end{aligned}$$

where $U(y) \in \gamma$ as f is an open bijection. Therefore, D' is a closed discrete subset of (Y, γ) . Finally, $\bigcup_{x \in D} \phi(x) = X$ implies that

$$Y = f(X) = f\left(\bigcup_{x \in D} \phi(x)\right) = \bigcup_{x \in D} f(\phi(x)) = \bigcup_{x \in D} \psi(f(x)) = \bigcup_{y \in D'} \psi(y)$$

as f connects ϕ with ψ . \square

Theorem 3.37. *Every connected I -space is a D -space.*

Proof. By condition, $X^d = \overline{X^d} = (X^d)^i$. Let us prove that $\bigcup_{x \in D} \phi(x) = X$ for any neighborhood assignment $\phi : X \rightarrow \tau$ on the TS (X, τ) , where $D = X^d$. Contrary: there is a neighborhood assignment $\phi : X \rightarrow \tau$ on the TS (X, τ) such that $\bigcup_{x \in D} \phi(x) \neq X$, i.e., $X \setminus \bigcup_{x \in D} \phi(x) \neq \emptyset$. Since the set $D = X^d \subset \bigcup_{x \in D} \phi(x)$, we have that

$$X \setminus \bigcup_{x \in D} \phi(x) = X^i \setminus \bigcup_{x \in D} \phi(x).$$

Therefore, it is not difficult to see, that

$$X^i \setminus \bigcup_{x \in D} \phi(x) = X^i \setminus \overline{\bigcup_{x \in D} \phi(x)}$$

and, hence, for the set $A = X \setminus \bigcup_{x \in D} \phi(x) \in \tau \cap \text{co } \tau$ we have: $\emptyset \neq A \neq X$, which is impossible as (X, τ) is connected. \square

Corollary 3.38. *Every connected TS (X, τ) , for which $X^i \in \mathcal{D}(X)$, is a D -space if it is nodec or submaximal.*

Proof. Follows directly from Theorem 1.6 and Corollary 1.7 in [5]. \square

Definition 3.39. A BS (X, τ_1, τ_2) will be called an (i, j) - D -space if for every i -neighborhood assignment $\phi_i : X \rightarrow \tau_i$ on the BS (X, τ_1, τ_2) there is a j -closed i -discrete subset $D \subseteq X$ such that $\bigcup_{x \in D} \phi_i(x) = X$.

It is clear that for a BS $(X, \tau_1 < \tau_2)$ we have:

$$(X, \tau_1, \tau_2) \text{ is a } 1\text{-}D\text{-space} \implies (X, \tau_1, \tau_2) \text{ is } (1, 2)\text{-}D\text{-space}$$

and

$$(X, \tau_1, \tau_2) \text{ is a } (2, 1)\text{-}D\text{-space} \implies (X, \tau_1, \tau_2) \text{ is a } 2\text{-}D\text{-space}.$$

By analogy with Theorem 3.36 one can prove

Theorem 3.40. *The following conditions are satisfied:*

- (1) *If for each i -neighborhood assignment $\phi_i : X \rightarrow \tau_i$ on the BS (X, τ_1, τ_2) there are i -neighborhood assignment $\psi_i : Y \rightarrow \gamma_i$ on the BS (Y, γ_1, γ_2) and a d -compression map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \gamma_1, \gamma_2)$ which connects ϕ_i with ψ_i , then (Y, γ_1, γ_2) is an (i, j) - D -space implies that (X, τ_1, τ_2) is also an (i, j) - D -space.*
- (2) *If for each i -neighborhood assignment $\psi_i : Y \rightarrow \gamma_i$ on the BS (Y, γ_1, γ_2) there are i -neighborhood assignment $\phi_i : X \rightarrow \tau_i$ on the BS (X, τ_1, τ_2) and a d -open bijection $f : (X, \tau_1, \tau_2) \rightarrow (Y, \gamma_1, \gamma_2)$ which connects ϕ_i with ψ_i , then (X, τ_1, τ_2) is an (i, j) - D -space implies that (Y, γ_1, γ_2) is also an (i, j) - D -space.*

Corollary 3.41. *The following conditions are satisfied:*

- (1) *If for a pair (ϕ_1, ϕ_2) , where $\phi_i : X \rightarrow \tau_i$ is an i -neighborhood assignment on the BS (X, τ_1, τ_2) , there are a pair (ψ_1, ψ_2) , where $\psi_i : Y \rightarrow \gamma_i$ is an i -neighborhood assignment on the BS (Y, γ_1, γ_2) , and a d -compression map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \gamma_1, \gamma_2)$ which connects (ϕ_1, ϕ_2) with (ψ_1, ψ_2) (i.e., f connects ϕ_1 with ψ_1 and ϕ_2 with ψ_2), then (Y, γ_1, γ_2) is a p - D -space implies that (X, τ_1, τ_2) is also a p - D -space.*
- (2) *If for a pair (ψ_1, ψ_2) , where $\psi_i : Y \rightarrow \gamma_i$ is an i -neighborhood assignment on the BS (Y, γ_1, γ_2) , there are a pair (ϕ_1, ϕ_2) , where $\phi_i : X \rightarrow \tau_i$ is an i -neighborhood assignment on the BS (X, τ_1, τ_2) , and a d -open bijection $f : (X, \tau_1, \tau_2) \rightarrow (Y, \gamma_1, \gamma_2)$ which connects (ϕ_1, ϕ_2) with (ψ_1, ψ_2) , then (X, τ_1, τ_2) is a p - D -space implies that (Y, γ_1, γ_2) is also a p - D -space.*

Theorem 3.42. *Every p -connected $(i, j)\mathcal{I}$ -space is an (i, j) - D -space.*

Proof. By condition, $X_j^d = \tau_j \text{ cl } X_j^d = (X_j^d)_i^i$. Let us prove that $\bigcup_{x \in D} \phi(x) = X$ for any i -neighborhood assignment $\phi_i : X \rightarrow \tau_i$ on the BS (X, τ_1, τ_2) , where $D = X_j^d$. Indeed, if there is an i -neighborhood assignment $\phi_i : X \rightarrow \tau_i$ on the BS (X, τ_1, τ_2) such that $\bigcup_{x \in D} \phi_i(x) \neq X$, then

$$\emptyset \neq X \setminus \bigcup_{x \in D} \phi_i(x) = X \setminus \tau_j \text{ cl } \bigcup_{x \in D} \phi_i(x)$$

as

$$X \setminus \bigcup_{x \in D} \phi_i(x) = X_j^i \setminus \bigcup_{x \in D} \phi_i(x).$$

Therefore, if $A = X \setminus \bigcup_{x \in D} \phi_i(x)$, then $A \in \tau_j \cap \text{co } \tau_i$, $\emptyset \neq A \neq X$ and so, by (c) of Theorem A in [18], (X, τ_1, τ_2) is not p -connected. \square

Corollary 3.43. *Every p -connected $p\mathcal{I}$ -space is a p - D -space.*

Corollary 3.44. *Every p -connected (i, j) -nodec BS (X, τ_1, τ_2) , for which $X_j^i \in i\text{-}D(X)$, is an (i, j) - D -space.*

Proof. Follows directly from (2) of Proposition 3.20. \square

Let $\Phi = \{\phi\}$ be the family of all neighborhood assignments on a TS (X, τ) . Then a binary relation \leq , defined on Φ in the manner as follows: $\phi_1, \phi_2 \in \Phi$, $\phi_1 \leq \phi_2$ if $\phi_1(x) \subseteq \phi_2(x)$ for each point $x \in X$, is a partial order on Φ . Evidently, this partial order is linear if for each pair $\phi_1, \phi_2 \in \Phi$ and each point $x \in X$ we have $\phi_1(x) \subseteq \phi_2(x)$ or $\phi_2(x) \subseteq \phi_1(x)$.

Theorem 3.45. *If for a TS (X, τ) the family Φ of all neighborhood assignments $\phi : X \rightarrow \tau$ on the TS (X, τ) is linearly ordered and there are topologies τ_1 and τ_2 on X such that $\sup(\tau_1, \tau_2) = \tau$ and (X, τ_1, τ_2) is a p - D -space, then (X, τ) is a D -space.*

Proof. Let $\phi : X \rightarrow \tau$ be any neighborhood assignment on the TS (X, τ) , where $\tau = \sup(\tau_1, \tau_2)$ for some topologies τ_1 and τ_2 on X . Without loss of generality we can suppose that for each point $x \in X$ the set $\phi(x)$ is basic open. Then $\phi(x) = U_1(x) \cap U_2(x)$, where $x \in U_i(x) \in \tau_i$. Let us define the neighborhood assignments $\phi_i : X \rightarrow \tau_i$ as follows: $\phi_i(x) = U_i(x)$ for each point $x \in X$. Since (X, τ_1, τ_2) is a p - D -space, for ϕ_1 there exists a 2-closed 1-discrete set D_1 and for ϕ_2 there exists a 1-closed 2-discrete set D_2 such that

$$\bigcup_{x \in D_1} \phi_1(x) = X = \bigcup_{x \in D_2} \phi_2(x).$$

It is clear, that $\phi_1, \phi_2 \in \Phi$ as $\tau_1 \cup \tau_2 \subset \tau$.

Furthermore, let $D = D_1 \cup D_2$. Because $\tau_1 \cup \tau_2 \subset \tau$, we have $D_1 \in \text{co } \tau_2 \subset \text{co } \tau$, $D_2 \in \text{co } \tau_1 \subset \text{co } \tau$ and, hence, $D \in \text{co } \tau$. Moreover, let $x \in D$ be any point, where

$$D = (D_1 \setminus D_2) \cup (D_1 \cap D_2) \cup (D_2 \setminus D_1).$$

If $x \in D_1 \cap D_2$, then $D_1 = (D_1)_1^i$ and $D_2 = (D_2)_2^i$ imply that there are neighborhoods $U(x) \in \tau_1$ and $V(x) \in \tau_2$ such that $U(x) \cap D_1 = \{x\} = V(x) \cap D_2$. Since $U(x) \in \tau_1 \subset \tau$, $V(x) \in \tau_2 \subset \tau$, we have that $W(x) = U(x) \cap V(x) \in \tau$ and $W(x) \cap D = \{x\}$ so that $D_1 \cap D_2 \subset D^i$. Now, if $x \in D_i \setminus D_j$, then there is a neighborhood $U(x) \in \tau_i \subset \tau$ such that $U(x) \cap D_i = \{x\}$. Let $V(x) = U(x) \setminus D_j$. Then $V(x) \in \tau_i \subset \tau$ and $V(x) \cap D = \{x\}$, so that $D_i \setminus D_j \subset D^i$ too.

Therefore, $D = D^i$ and it remains to prove only that $\bigcup_{x \in D} \phi(x) = X$. But

$$\bigcup_{x \in D} \phi(x) = \bigcup_{x \in D} (\phi_1(x) \cap \phi_2(x)).$$

Since $\phi_1, \phi_2 \in \Phi$ and Φ is linearly ordered, we have that $\phi_1(x) \subseteq \phi_2(x)$ or $\phi_2(x) \subseteq \phi_1(x)$ for each point $x \in D$. Let, for example, $\phi_2(x) \subseteq \phi_1(x)$ for each point $x \in D$. Then

$$\bigcup_{x \in D} \phi(x) = \bigcup_{x \in D} \phi_2(x) = X$$

and, consequently, (X, τ) is a D -space. \square

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